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Optimal Asset Allocation for Commodity Sovereign Wealth Funds*

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Abstract

This paper solves a dynamic asset allocation problem for a commodity sovereign wealth fund under incomplete markets. We calibrate the model using data from three countries: Norway, UAE and Chile. In our benchmark calibration for Norway, we find that the fund's manager should initially invest all her wealth to stock and reduce this fraction gradually over time. We find that the solution is particularly sensitive to the assumption about the volatility of commodity prices. The solution for Chile implies that for relatively high risk aversion coefficients the manager should start at a small fraction of her wealth to increase later over the life cycle of the fund.

JEL Classification: *E21, G11.*

Keywords: *Dynamic asset allocation, portfolio management, sovereign wealth fund, income risk.*

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1 Introduction

Sovereign wealth funds (SWFs hereafter) are institutional investors that engage in long-run policies with the objective of ensure gradual transfers of wealth across generations in order to maximize long-run expected returns. Most SWFs' source of income comes from the commodity sales revenues and/or the accumulation of foreign exchange reserves. Although these investment funds have existed for decades, there has been a significantly increase of SWFs since 2000. As of 2013 there are 55 different commodity based SWFs in the world administering US$4 trillion in assets, corresponding to US$1,211 billion more than the estimated size of hedge funds worldwide, and to 3 percent of the global investment industry (SWF Institute, 2014 and Research, 2015). As prominent actors in the global asset market, the SWFs are expected to enhance the performance of the funds through optimal management of portfolio investments.

Given their importance in the global asset markets, this paper investigates, from a normative perspective, the optimal portfolio choice of a commodity SWF with a long-term investment horizon. To do so, we set up and solve an otherwise standard dynamic asset allocation problem following the contributions of Bodie et al. (1992) and Campbell and Viceira (2002). In particular, we focus on the following problem faced by the fund's manager: given his preferences for risk, how much of the fund's wealth should be invested in a risk-free asset and how much should be invested in a risky asset (stocks) over a long investment horizon. At this point we abstract from any political and fiscal considerations, but we will include them in future research. What distinguishes a commodity SWFs from other institutional investors is the source of their income and the correlated income risk with the asset market. In general, the revenues from commodities are highly volatile, decreasing over time as the resource is depleted and partially correlated with the stock market. To gain analytical tractability and insight on the mechanisms behind the asset allocation choices made by the fund's manager we first solve the model under the assumption that all the income risk is perfectly spanned by the stock market and hence markets are said to be complete. Following Munk and Sørensen (2010), we show that under complete markets the main determinants of the optimal investment strategy over time are the dynamic behavior of the commodity-wealth to financial-wealth ratio, the fund's manager level of risk aversion and the income volatility.

We later abandon the assumption of complete markets and solve the model using global approximation methods. We study the optimal asset allocation, and solve for both unconstrained (where agents can borrow/lend assets) and short-sales constrained problems by calibrating the model in such a way that it matches salient features of the Norwegian SWF. We use the calibrated model to draw quantitative results. The predictions of the unconstrained problem indicate that the Norwegian Petroleum Fund should leverage its investment in the risky asset in order to allocate more than its total financial wealth in stocks during its first year of operation and then gradually decrease its position to a long-run fraction of 60 percent after 30 years. When we impose the constraint on short-sales, the model predicts that the fund should keep all its financial wealth invested in the risky asset for the first five years, and then start decreasing the investment share on stock gradually to a long-run share of 60 percent in about 40 years. Both implications are explained by an initially large oil-wealth to financial-wealth ratio that allows the fund's manager to take large risk positions when the natural resource is far from being depleted. To take into account the effects of high and low volatilities in the price of oil we perform a sensitivity experiment. We find that changes in the volatility of the oil price can substantially affect the size of the hedging demand that lead to differences of up to 30 percent in the investment share of stocks when moving from a low to a high variability level of prices.

We conduct a similar analysis for the commodity SWFs of Chile and UAE. Given that UAE has larger levels of oil reserves and similar current financial wealth, the optimal constrained allocation suggests the fund's manager to invest a larger share of the risky asset throughout the investment horizon. For the case of Chile, we find that for reasonable values of the risk aversion and given the observed cross-correlation between the price of copper and the stock returns there are scenarios for which the SWF should invest only a small fraction of its financial wealth in the risky asset at the beginning of the investment horizon and then increase it substantially as the resource is depleted.

Our setup builds on the dynamic asset allocation framework of Merton (1971), Campbell and Viceira (2002), Stoikov and Zariphopoulou (2005) and Chacko and Viceira (2005) extended to include the stream of risky labor income that the investor receives over time. Svensson and Werner (1993) and Henderson (2005) have derived closed form solutions for such a problem by assuming that investors have preferences that can be represented by a negative exponential utility function and the presence of imperfect correlation between income growth and stock returns. More recently, Munk and Sørensen (2010) restore to numerical methods to provide a solution to an asset allocation problem with stochastic interest rate and labor income under incomplete markets, and find that the shocks to labor income and interest rate have considerable effects on the optimal investment decisions.

Our work also relates to a number of recent contributions that study the asset allocation problem faced by SWFs. Dyck and Morse (2011) use panel data to examine the objectives driving SWF investments. Scherer (2011) formulates a portfolio allocation problem where the fund's manager must choose how much to invest in risky assets under a mean-variance framework where his payoff is determined not just by the funds wealth but the entire government's economic wealth. He finds that the optimal investment on risky asset is a function of financial-wealth to oil-wealth ratio, assuming constant government budget relative to financial wealth. van den Bremer et al. (2016) study the investment implications for a SWF in a model for oil exporter countries by combing theories of portfolio allocation and optimal extraction rate. Using an approximated solution of the model under the assumption of incomplete markets, they find that the policy maker should consider the below-ground wealth in his optimal investment strategy for the SWF. Although providing a natural framework to perform different qualitative experiments, their model is not able to match the observed data.

The remainder of the paper proceeds as follows. In Section 2 we provide some stylized facts about the three SWFs studied in this paper. Section 3 describes the asset allocation problem for a commodity SWF and provides a closed form solution for the optimal portfolio choice under complete markets. In Section 4 we relax the assumption of complete markets and solve the dynamic asset allocation problem using the numerical method proposed in Munk and Sørensen (2010). The model is calibrated to mimic some of the characteristics of the Norwegian SWF. We also perform a sensitivity analysis to different levels of oil price volatility. Section 4 also compares the optimal investment strategies of the SWFs in Norway, UAE and Chile. Finally, Section 5 concludes.

2 Commodity prices and SWFs in different countries

In this section, we describe the data used in this study and characterize some statistical moments among commodities and stock prices. Then, we describe financial and technological conditions for the SWFs of Norway, UAE and Chile.

Data. We use annual data on commodities and stock prices for the period 1900-2013. The price of oil corresponds to the West Texas Intermediate (WTI) reference price, while for copper we use the high grade copper price. For the stock market we use the S&P 500 composite price index. The commodity and stock prices are collected from Global Financial Data (2014). Nominal prices are deflated using the Consumer Price Index reported by the U.S. Bureau of Labor Statistics (2014). The base year is 2010 when computing real values. Commodity production for Norway and the UAE are taken from the U.S. Energy Information Administration (2014), while the reserves of crude oil are obtained from BP (2014). For the Chilean SWF we use cooper production data from the Chilean Copper Commission (2014), while the data on copper reserves is collected from the U.S. Geological Survey (2014). The data on financial wealth for Norway, the UAE and Chile are obtained from Norges Bank

Table 1: Empirical moments: commodity and stock prices

Moments	1900-2013	1900-1941	1942-1973	1974-2013
Stock price				
$\mathbb{E}\left(\Delta \log S_t\right)$	0.018	-0.002	0.041	0.020
$\text{std}\left(\Delta \log S_t\right)$	0.195	0.229	0.162	0.184
Oil prices				
$\mathbb{E}\left(\Delta \log P_t^{oil}\right)$	0.005	-0.009	-0.020	0.040
$\text{std}\left(\Delta \log P_t^{oil}\right)$	0.247	0.240	0.042	0.336
Copper price				
$\mathbb{E}\left(\Delta \log P_t^{copper}\right)$	-0.004	-0.027	0.010	0.008
$\text{std}\left(\Delta \log P_t^{copper}\right)$	0.240	0.235	0.143	0.303
Cross Correlations				
$\text{corr}\left(\Delta \log S_t, \Delta \log P_t^{oil}\right)$	0.039	0.121	0.133	-0.034
$\text{corr}\left(\Delta \log S_t, \Delta \log P_t^{copper}\right)$	0.321	0.605	0.075	0.137

Notes: The table reports empirical moments for commodity and stock prices using annual data. The stock price corresponds to the S&P 500 composite price index, oil price is measured by the WTI reference price and the copper price is collected from the high grade copper price. Commodity and stock prices are from the Global Financial Data (2014). Nominal prices are deflated by the U.S. CPI deflator from U.S. Bureau of Labor Statistics (2014).

Investment Management (2013), SWF Institute (2014) and Chilean Copper Commission (2014), respectively. Finally, the GDP data is taken from World Bank (2014).

Table 1 summarizes some empirical moments for the real price of oil, copper and stocks for four different subsamples. The calculations reveal some interesting facts for commodity prices regarding their price volatility and correlation with the stock market over long periods of time. In particular, we find that the volatility of variations in the oil price increases from 4.2 percent per year in the post war period (1942-1973) to 33.6 percent per annum during 1974-2013. Similarly, the correlation between variations in the price of oil and stock returns goes from a positive value of 13.3 percent per year to a mildly negative value of -3.3 percent per year in the same subsamples. In what follows, we will calibrate our benchmark model using the moments for the whole sample period (1900–2013) and later on perform a series of sensitivity analysis using the volatility of commodity prices found in the different subsamples.

Three commodity based SWFs. In this paper we study three commodity funds: Norway, the UAE and Chile. Figure 1 provides some information regarding their current and expected levels of wealth. Panel (a) plots the real value of their asset holdings in 2013. Norway is one of the world largest established sovereign fund with income source corresponding to oil revenues. For the year 2013, the Norwegian Government Pension Fund Global (GPFG) held US$770 billion in assets, nearly two and half times the Norwegian real GDP. In practice, the Norway's Ministry of Finance owns the fund and determines a investment strategy in

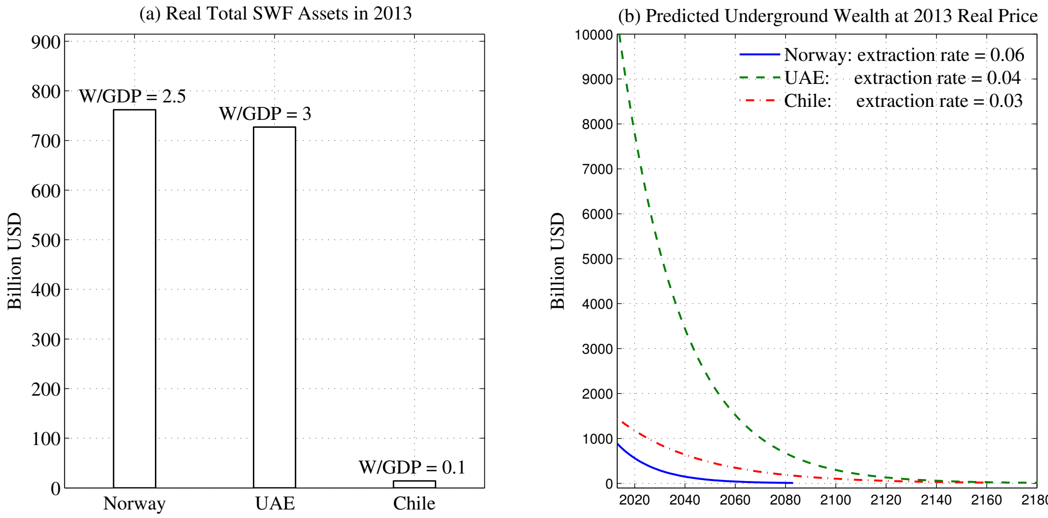

Figure 1: Current financial and underground wealth for different countries.

Notes: Panel (a) plots the real asset holding in 2013 for the SWF of Norway, the UAE and Chile. The financial wealth GDP ratio in 2013 for each country appears at the top of each bar. Panel (b) predicts the underground commodity wealth for Norway, the UAE and Chile since 2013. The wealth paths are computed assuming constant real prices of commodities as of 2013, zero exploration and constant extraction rate at the 2013 level for the three countries. WTI oil price and real high grade copper price are collected from Global Financial Data (2014). Nominal prices are deflated using the U.S. CPI deflator from U.S. Bureau of Labor Statistics (2014). The production of oil in Norway and the UAE in 2013 are from U.S. Energy Information Administration (2014). Chilean copper production in 2013 is from Chilean Copper Commission (2014). The crude oil reserve values are obtained from BP (2014). The Chilean reserve of copper is from U.S. Geological Survey (2014). The financial wealth in Norway is from Norges Bank Investment Management (2013). The financial wealth data for the UAE is from SWF Institute (2014). The financial wealth of Chile is from Ministry of Finance Gobierno de Chile (2013). The GDP data of three countries is from World Bank (2014).

2009–2013 with a constant of 60 percent equity, 40 percent bonds[1]. According to the Ministry of Finance Norway (2015), the Norwegian government commits to a budgetary rule that is 4 percent of the fund asset each year. Furthermore, the UAE SWF is another world largest sovereign fund based on oil revenues. The Abu Dhabi Investment Authority (ADIA), the UAE SWF, held US$720 billion USD in assets, around three times the UAE's real GDP in 2013. On the other hand, The Economic and Social Stabilization Fund is the SWF in Chile which is a relative young fund based on the copper revenue. Their 2013 asset holdings amounted to US$14 billion, a 10 percent of Chile's real GDP.

Panel (b) in Figure 1 depicts the value of underground reserves for the three commodity SWFs by assuming zero further exploration after 2013[2], and a constant production rate of the remaining reserves at each point in time. In particular, we assume that the future extraction-reserve rate remains constant at its 2013 level which was 6 percent for Norway, 4 percent for UAE and 3 percent for Chile. Therefore, using the effective levels of reserves in 2013 we

[1]Now it is 60 percent equity, 35 percent bonds and 5 percent real estate (Norges Bank Investment Management 2013, 2015)

[2]The assumption of zero exploration may underestimate their real financial wealth. Future analysis should include a time-varying production rate with possible resource exploration.

predict the trajectories of underground reserves in Norway, the UAE and Chile. In order to make the numbers comparable we value the predicted reserves of oil and copper at 2013 real prices and assuming no price changes thereon[3]. Moreover, we also compute the time of resource depletion[4] as the terminal date with positive underground wealth for each country[5]. Given their large initial reserves of oil, our calculation indicate that the 2013 underground wealth for the UAE amounts to US$10 thousand billion, and the resource will be completely depleted in 170 years. Norway, starting with US$900 billion in underground oil wealth will reach resource depletion in around 70 years. Finally, given the low rate of extraction, Chile will deplete their reserves of copper in 150 years. We use these values when computing optimal portfolio allocations in Section 4.2.

3 An asset allocation model for a SWF manager

In this section, we describe the problem faced by a commodity SWF manager. At this point we abstain from any fiscal and political considerations and instead we focus on the optimal asset allocation made by the fund's manager in order to ensure a stable stream of transfers to the government when it is known *a priori* that the revenues of the fund are decreasing over time due to the depletion of the resource. To do so, we first assume that markets are complete and derive a closed form solution for the optimal allocations that will give us some intuition about the mechanism and driving forces behind the model. We later study the allocation implications under the assumption of market incompleteness. Our framework build on the early contributions of Merton (1971) and assumes that time in the economy evolves continuously and that all the uncertainty faced by the investor can be represented by a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with associated filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$.

3.1 Description of the model

Asset returns. Let us assume that the SWF manager has a costless access to two tradable assets. A risk free bond with instantaneous return r, and a risky asset (stocks) with instantaneous return $r + \psi$, where ψ denotes the expected excess return of stocks over the

[3]The 2013 reserves of crude oil in Norway and the UAE were 8.7 billion barrels and 101 billion barrels, respectively. The 2013 reserves of copper in Chile were 190 million metric tons. The real price of oil in 2013 was US$102 per barrel while the real price of copper was US$7651 USD per metric tons.

[4]With constant extraction rates, production quantities of oil and copper decline infinitely. In the calibration, we compute the time of depletion corresponding to date when the terminal production revenues are less or equal to US$100 million. The terminal production revenue is computed as the product of terminal extraction and real commodity price in 2013.

[5]The underground wealth is assumed to be zero after the depletion time.

risk free bond. The price of the stock S_t is assumed to evolve according to:

$$\frac{dS_t}{S_t} = (r + \psi)\, dt + \sigma_S dz_{St}, \tag{1}$$

where σ_S is the volatility of the stock price and z_{St} is a standard Brownian motion.

Commodity revenues. Assuming zero exploration (and discoveries) of new reserves, the availability of the natural resource q_t decreases at a constant extraction-reserve rate $\alpha_q > 0$ for $\hat{T}$ periods:

$$\frac{dq_t}{q_t} = -\alpha_q dt. \tag{2}$$

The price of the commodity is assumed to follow a geometric Brownian motion with a drift α_p and volatility σ_y:

$$\frac{dp_t}{p_t} = \alpha_p dt + \sigma_y \left(\rho_{yS} dz_{St} + \sqrt{1 - \rho_{yS}^2} dz_{yt} \right) \tag{3}$$

where z_{yt} is a standard Brownian motion and $\rho_{yS} \in [-1, 1]$ denotes the correlation between variations in the commodity price and the stock returns. For simplicity, we assume that z_{yt} and z_{St} are independent. Therefore, and assuming zero cost of production, the disposable income for the SWF is given by $y_t = p_t q_t$. Using Itô's Lemma, the dynamics of the fund's revenues follow:

$$\frac{dy_t}{y_t} = \alpha dt + \sigma_y \left(\rho_{yS} dz_{St} + \sqrt{1 - \rho_{yS}^2} dz_{yt} \right) \quad \forall t \leq \hat{T}, \tag{4}$$

where $\alpha = (\alpha_p - \alpha_q)$ represents the expected income growth. In what follows, we will assume that the fund's income is equal to zero after the full depletion of the exhaustible resource, that is, $y_t = 0$ for all $\hat{T} \leq t \leq T$ where $T > \hat{T}$ corresponds to the investment horizon of the fund's manager. Given our set of assumptions, ρ_{yS} also denotes the instantaneous correlation between the fund's income growth and the stock returns. When $\rho_{yS} = 1$, the income risk is fully spanned by the tradable stock. We refer to this scenario as the complete market case.

The SWF manager's decision problem. Following Campbell and Viceira (2002) and Munk and Sørensen (2010), we assume that the manager of the SWF faces the classical consumption-investment problem. Starting from instant $t$and given the current level of wealth, W_t, and prevailing prices in the economy, the manager chooses the paths of consumption and portfolio allocation, $\{c_u, \theta_{Su}\}_{u=t}^{T}$, that provide him the maximum level of life-time utility, $J(W,y,t)$. The latter is also known as the value function. Formally, he solves:

$$J(W,y,t) = \max_{\{c_u,\theta_{Su}\}_{u=t}^{T}} \mathbb{E}_t\left[\int_t^T e^{-\delta t}U(c_u)\,du + \epsilon e^{-\delta(T-t)}U(W_T)\right] \tag{5}$$

subject to the budget constraint:

$$dW_t = (rW_t + \psi\theta_{St} + y_t - c_t)\,dt + \sigma_S\theta_{St}dz_{St} \tag{6}$$

where $\delta > 0$ is the discount rate, $U(\cdot)$ is an increasing and concave utility function and $J(W,y,T) = \epsilon U(W_T)$ is a given terminal condition with parameter ϵ for terminal level of wealth W_T. The portfolio allocation θ_{St} denotes the amount, in nominal terms, invested at time t in the risky asset. Similarly, the consumption choice should be understood as transfers that the fund's manager makes to the central government which will eventually become part of the government's budget constraint. Therefore, in what follows we will refer to c_t as consumption or government transfers at time t interchangeably.

Finally, we assume that the fund's manager preferences are described by a standard, time separable, power instantaneous utility function over consumption:

$$U(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma} \tag{7}$$

where $\gamma > 0$ is the coefficient or relative risk aversion and $1/\gamma$ the intertemporal elasticity of substitution of consumption.

3.2 Optimal investment strategies

In order to compute the optimal investment strategy for the SWF, we will use the dynamic programming approach originally proposed in Merton (1971) for the case of continuous-time models. At every point in time t, the fund's manager gathers information on his financial wealth W_t and commodity revenue y_t to make his optimal choices. With these values for the state variables of the problem, the dynamic programming equation, also known as the Hamilton-Jacobi-Bellman (HJB) equation, is given by:

$$\delta J\left(W_t, y_t, t\right) = \max_{c_t, \theta_{St}} U\left(c_t\right) + J_t\left(W_t, y_t, t\right) + J_W\left(W_t, y_t, t\right)\left[rW_t + \psi\theta_{St} + y_t - c_t\right]$$
$$+ \frac{1}{2} J_{WW}\left(W_t, y_t, t\right)\sigma_S^2\theta_{St}^2 + \alpha y J_y\left(W_t, y_t, t\right)$$
$$+ \frac{1}{2} J_{yy}\left(W_t, y_t, t\right) y_t^2\sigma_y^2 + J_{Wy}\left(W_t, y_t, t\right) y_t\theta_{St}\sigma_S\sigma_y\rho_{yS} \quad (8)$$

with terminal condition

$$J\left(W_T, y_T, T\right) = \epsilon U\left(W_T\right) = \frac{\epsilon W_T^{1-\gamma}}{1-\gamma}, \quad (9)$$

where J_i denotes the first order partial derivative the value function with respect to the state variable i, and J_{ij} the second order derivative with respect to the state variables i and j.

Let $\pi_{St} = \frac{\theta_{St}}{W_t}$ denote the fraction of wealth invested in the risky asset. Given the manager preferences, the first order conditions for an interior solution are given by:

$$c_t = J_W^{-\frac{1}{\gamma}} \quad (10)$$
$$\pi_{St} = \underbrace{\frac{1}{-\frac{W_t J_{WW}}{J_W}}\frac{\psi}{\sigma_S^2}}_{\text{Myopic-demand}} + \underbrace{\frac{y_t J_{Wy}}{J_W}\frac{1}{-\frac{W_t J_{WW}}{J_W}}\frac{\sigma_y\rho_{yS}}{\sigma_S}}_{\text{Hedging-demand}}. \quad (11)$$

Equation (10) defines the optimal level of transfers from the fund to the government such that at the maximum an extra unit of consumption should be as valuable, to the decision maker, as an extra unit of wealth that could be used to finance future transfers.

Equation (11) determines the optimal portfolio allocation to the risky asset as the combination of two components. The first term is usually referred to as the *myopic portfolio rule* and corresponds to the investment strategy that an investor with a short investment horizon will follow. It corresponds to the standard investment recommendation from the mean-variance analysis of Markowitz (1952) that suggest that the optimal fraction of wealth invested in the risky asset should be proportional to the asset's risk premium over the risk free asset, ψ, and inversely proportional to the asset's volatility, σ_S, and the manager's risk aversion, the latter measured by the curvature of the value function, $-\frac{WJ_{WW}}{J_W}$.

The second term is the intertemporal hedging component against commodity income risk and represents the extra demand required by an investor with a long horizon investment. In particular, the hedging demand will be determined by the volatility of income, σ_y, and its correlation with the stock returns, ρ_{yS}. The term $\frac{\sigma_y\rho_{yS}}{\sigma_S}$ can be seen as the regression coefficient of commodity revenue innovations on stock return innovations. It is also a function of the investor's risk aversion and his aversion to income revenue risk, measured by $\frac{yJ_{Wy}}{J_W}$.

Importantly, this component suggests that the manager should increase his holding of the risky asset for increased levels of aversion to income risk and whenever it returns covary negatively with the commodity based income growth.

3.3 The case of complete markets

To obtain some intuition about the determinants of the demand for risky assets we first assume that markets are complete. Under complete markets, the fund's income is spanned by the risky asset and hence it is possible to derive a closed form solution following Munk and Sørensen (2010). To do so, we first compute the certainty equivalent value of the stream of future oil income, in other words, the commodity wealth. Therefore we introduce $\mathbb{Q}$ to be the risk-neutral probability measure and let $O(y_t, t)$ denote the time t net present value of all future commodity revenues until depletion time $\hat{T}$:

$$O\left(y_t, t\right) \quad = \quad E^{\mathbb{Q}}_{y,t}\left[\int_t^{\hat{T}} e^{-r(s-t)} y_s ds\right],$$

i.e. the value that the SWF would obtain if it was possible to sell all the future revenues from oil in the financial markets. The next proposition provides an expression to compute the commodity wealth:

Proposition 3.1 (Commodity wealth and income multiplier)**.** *Under complete markets, i.e. for $\rho_{ys} = \pm 1$, the commodity wealth of the SWF at time t is given by:*

$$O\left(y_t, t\right) = y\mathbf{1}_{\{t \leq \hat{T}\}} M\left(t\right) \tag{12}$$

where $M\left(t\right)$ is called the commodity income multiplier and it takes the form:

$$M\left(t\right) = \frac{1}{r - \alpha + \sigma_y \rho_{yS} \frac{\psi}{\sigma_S}} \left(1 - e^{-\left(r - \alpha + \sigma_y \rho_{yS} \frac{\psi}{\sigma_S}\right)\left(\hat{T} - t\right)}\right) \tag{13}$$

for $\left(r - \alpha + \sigma_y \rho_{yS} \frac{\psi}{\sigma_S}\right) \neq 0$ and all $t \leq \hat{T}$. The indicator function $\mathbf{1}_{\{t \leq \hat{T}\}}$ reflects the fact after the resource has been depleted the commodity income is zero, i.e. $O\left(y_t, t\right) = 0$ for all $t = \hat{T}, ..., T$.

Proof. See Appendix A. □

To gain some intuition about the role played by the commodity income multiplier, Figure 2 plots how the commodity wealth as a fraction of current commodity income, $M\left(t\right)$, as a function of the depletion horizon, $\left(\hat{T} - t\right)$, for different expected income growth rates, α,

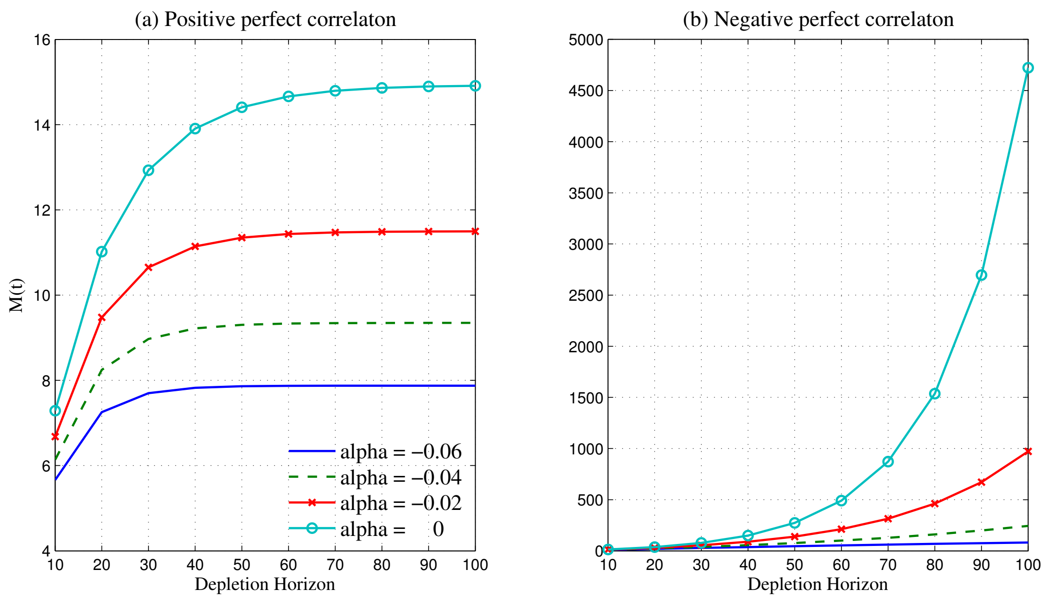

Figure 2: Commodity income multiplier $M(t)$

Notes: Panels (a) and (b) plot the commodity income multiplier $M(t)$ for different depletion time horizons $\left(\hat{T}-t\right)$ and expected income return α under complete markets. Panel (a) reports the multiplier behavior when income growth and stock returns are perfectly correlated in the same direction, $\rho_{yS}=1$, and Panel (b) when the direction is opposite, $\rho_{yS}=-1$. The remaining parameters are set to $r=0.006$, $\psi=0.05$, $\sigma_s=0.2$.

and both perfect positive and negative income-stock correlation. The remaining parameters are set to reasonable values.

Panel (a) in Figure 2 plots the commodity income multiplier when income growth and stock returns are perfectly and positively correlated. Panel (b) plots the multiplier with perfect negative commodity income and stock correlation. Our calculations indicate that the income multiplier is increasing in the expected income growth rate α and the depletion horizon $\hat{T}-t$. The commodity income multiplier is high when resource depletion is far away: concave for a positive correlation, and convex for a negative correlation. For example, if income and stocks exhibit a perfect positive correlation, then a fund's manager that expects the natural resource to be depleted in 70 years from now will value his current commodity wealth to be around 11 times the actual level of revenues. However, in the case of perfect negative correlation, the fund's manager could trade his stream of revenue in the market in order to perfectly insure against negative fluctuations in the stock market. This leads to a higher valuation of his current commodity wealth (more than 1000 times his current income) when depletion of the resource is 70 years ahead.

To solve the closed-form optimal investment strategy we still need to know the form of the value function $J(W_t, y_t, t)$ that solves the partial differential equation described by the

HJB Equation (8). We use a *guess-and-verify* method to find the exact form of the value function. Following Munk and Sørensen (2010) we conjecture that the value function takes the form:

$$J\left(W_t, y_t, t\right) = \frac{1}{1-\gamma} g\left(t\right)^{\gamma} \left(W_t + O\left(y_t, t\right)\right)^{1-\gamma} \tag{14}$$

where $g(t)$ is generally unknown. To solve for the function $g(t)$ we insert our guess, its derivatives, and the first order conditions (10) and (11) into the HJB Equation (8) and derive find that the functions: (A full derivation can be found in Appendix B):

$$g\left(t\right) = \frac{1}{A}\left[1 - e^{-A(T-t)}\right] + \epsilon^{\frac{1}{\gamma}} e^{-A(T-t)}$$

$$A = \frac{1}{\gamma}\left[\delta - r\left(1-\gamma\right) - \frac{1-\gamma}{2\gamma^2}\left(\frac{\psi}{\sigma_S}\right)^2\right]$$

satisfy the dynamic programming equation for all possible values of W_t and $O\left(y_t, t\right)$ and $t = 1, \ldots, T$. Hence, the guess is correct and can be used to find exact expressions for the optimal level of transfers and the optimal asset allocation. In particular, Equation (10) indicates that optimal consumption is proportional to total wealth:

$$c(W_t, y_t, t) = \frac{1}{g\left(t\right)}\left(W_t + O(y_t, t)\right). \tag{15}$$

Alternatively, the consumption to financial wealth ratio is given by:

$$\frac{c_t}{W_t} = \frac{1}{g\left(t\right)}\left(1 + \frac{O_t}{W_t}\right) = \frac{1}{g\left(t\right)}\left(1 + \frac{y_t}{W_t}\mathbf{1}_{\{t \leq \hat{T}\}} M\left(t\right)\right) \tag{16}$$

suggesting that, provided that $g\left(t\right) > 0$, consumption is an increasing function of the ratio of commodity wealth to financial wealth which changes over the life cycle of the fund. In fact, after resource depletion, i.e. for $\hat{T} \leq t \leq T$, the commodity wealth to financial wealth ratio is zero and hence consumption only depends proportionally of financial wealth, with proportionality factor given by the inverse of $g\left(t\right)$.

On the other hand, the optimal investment share on the risky asset is given by:

$$\pi_S(W_t, y_t, t) = \frac{1}{\gamma}\frac{\psi}{\sigma_S^2} + \frac{O\left(y_t, t\right)}{W_t}\frac{1}{\sigma_S}\left(\frac{1}{\gamma}\frac{\psi}{\sigma_S} - \sigma_y \rho_{yS}\right). \tag{17}$$

The first term, $\frac{1}{\gamma}\frac{\psi}{\sigma_S^2}$, is the myopic investment rule. Clearly, investors will increase their holdings of risky asset if they are less risk averse. The second term, $\frac{O_t}{W_t}\frac{1}{\gamma}\frac{\psi}{\sigma_{S^2}}$, adjusts the optimal allocation due to the deterministic effects of having the commodity income, while the last term defines the hedging demand due to income risks and their (perfect) correlation to fluctuations in stock returns. If $\rho_{yS} = 1$, the hedging demand is negative: the manager should take less risk, as the income substitutes stock holdings. On the contrary, if $\rho_{yS} = -1$ income composites stock holdings and the hedging demand is positive. Therefore the manager should increase his position in the risky asset.

To better understand the effects of oil revenues on the optimal investment decision, let us consider the case of a deterministic flow of oil income, i.e. $\sigma_y = 0$. In this case, the investor's allocation is determined by the time path of the commodity wealth to financial wealth ratio:

$$\pi_S(W_t, y_t, t) = \frac{1}{\gamma}\frac{\psi}{\sigma_S^2}\left(1 + \frac{O\left(y_t, t\right)}{W_t}\right).$$

Hence, for a decreasing trajectory, the investor will take more risk initially and will eventually approach the myopic rule as the resource is depleted and the fraction $\frac{O_t}{W_t}$ goes to zero. In general, SWF with large initial $\frac{O_t}{W_t}$ ratio will take more risk. Now, let us suppose that the commodity income is risky. Now whether the investment strategy approaches the myopic rule from above or below will depend on the sign of the term in brackets in Equation (17). If $\left(\frac{1}{\gamma}\frac{\psi}{\sigma_S} - \sigma_y\rho_{yS}\right) < 0$ then the share of the risky asset decreasing over time. In the opposite case, the share is initially lower but increasing over time.

A numerical example. To illustrate some features of the solution under complete markets we perform some numerical simulations. Figure 3 plots the manager's yearly optimal allocation in risky assets over the SWF's investment horizon for two income volatility scenarios and fixing the remaining parameters at economically reasonable values. In particular, we assume an expected oil income growth $\alpha = -5\%$, a risk-free return of $r = 0.6\%$, a stock volatility of $\sigma_s = 20\%$, a relative risk aversion of $\gamma = 2$, and a discount rate of $\delta = 3\%$. We further assume that the reserves of the natural are depleted in 100 years ($\hat{T} = 100$), and that the initial ratio of financial-wealth to commodity-income is $\frac{W_0}{y_0} = 13$. We simulate 10000 samples of income, y_t, and stock prices, S_t, each of 100 observations, and compute the optimal investment share using Equations (12) and (17).

The results in Figure 3 correspond to the average value over the 10000 simulations for each point in time. The first scenario is represented by the solid line and it corresponds to the case where there is no uncertainty in the commodity income, i.e. when $\sigma_y = 0$. It is possible to show by rearranging Equation (17), that in this case the optimal investment in stocks relative to total wealth is constant and equal to $\frac{1}{\gamma}\frac{\psi}{\sigma_S^2} = 60\%$. In terms of financial wealth and given our initial wealth to income ratio, the average optimal investment on stocks goes from 143 percent at the beginning of the investment horizon and decreases to the myopic rule of 60 percent at depletion time $\hat{T} = 100$.

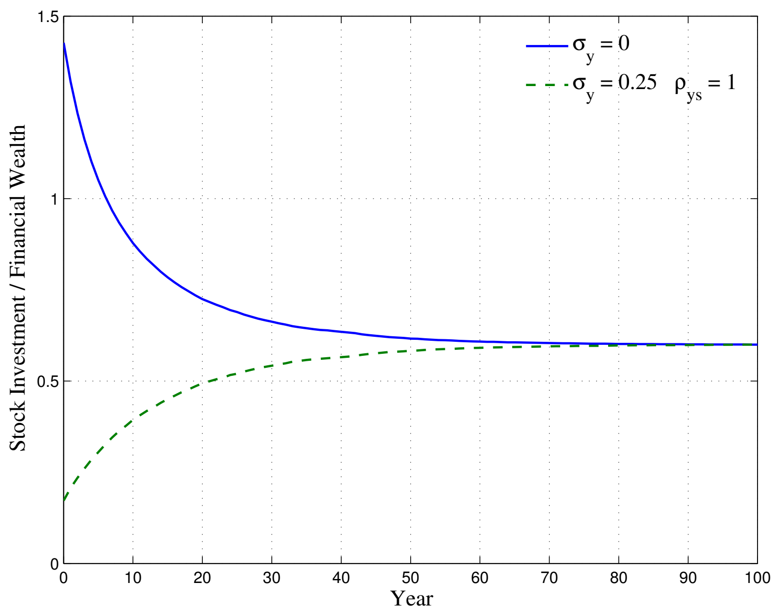

Figure 3: Optimal asset allocation under complete market: deterministic and stochastic income

Notes: The figure plots the optimal fraction of financial wealth invested in the stock under complete market, π_{St}. We report the mean over 10000 simulations. Each of the simulated sample paths for optimal investment share is computed using Equation (17). The solid line represents the expected investment share on stock with $\sigma_y = 0$. The dash line represents the expected investment share on stock with $\sigma_y = 0.25$ and $\rho_{ys} = 1$. The remaining parameters are set to be $\alpha = -0.05$, $r = 0.006$, $\psi = 0.05$, $\sigma_s = 0.2$, $\gamma = 2$, $\delta = 0.03$, $\hat{T} = 100$ and $\frac{W_0}{y_0} = 13$.

The second scenario, represented by the dash line, considers the case of stochastic oil income with $\sigma_y = 0.25$ and $\rho_{yS} = 1$. In this case, the investor receives an additional source of risk from oil revenues that needs to be weighted when making investment decisions. Since oil income growth and stock returns are assumed to be perfect and positive correlated, the fund's manager hedges against oil income fluctuations by reducing his demand for stocks. Our simulation results suggest that the optimal investment share should increase gradually and monotonically over time from 17 percent to the 60 percent long run share.

4 Quantitative predictions

This section explores the quantitative predictions of the model introduced in Section 3. We begin our analysis by calibrating the parameters of the model to reflect relevant and salient features of the Norwegian SWF. In particular, we discuss the implications for the optimal investment strategies under incomplete markets and short-sale restrictions. Finally, we compare our results with two other SWFs, the UAE and the Chilean.

4.1 Model calibration

Table 3 summarizes the parameter values for the case of the Norwegian SWF. Time is measured in years and parameters should be interpreted accordingly. The sample period used in the calibration is 1900-2013. The expected growth of oil prices, α_p, is set to be 1 percent as suggested by the mean growth rate of oil price. As discussed previously, we assume a constant extraction rate equal to its 2013 level of 6 percent, which implies an expected growth rate of oil income, α, of -0.05 (0.01-0.06). The volatility of income growth, σ_y, is computed as the standard deviation of oil price growth and the covariance parameter ρ_{yS} as the correlation between oil price growth and stock returns. The expected equity returns, $\psi + r$, is set to 0.055 and their volatility parameter, σ_S, to 0.2. The parameter values for the stock returns are consistent with the magnitude of the equity premium reported in Beeler and Campbell (2012) for similar sample period.

The fund's manager preferences are calibrated as follows. We set his risk aversion parameter, γ, to be 2, the discount factor, δ, equal to 0.03 and the risk free rate of 0.6 percent. Together with the values for the stock returns, our calibration of preferences ensure a myopic investment rule of about 60 percent. Finally, we solve the dynamic asset allocation problem assuming an investment horizon larger than the time to depletion. The latter is set to be $\hat{T} = 70$ for Norway.

Table 2: Benchmark calibration for the Norwegian SWF

Description	Parameter	Value
Drift for oil price	α_p	0.010
Extraction rate	α_q	0.060
Drift for oil income	α	-0.050
Risk free rate	r	0.006
Excess return	ψ	0.049
Income volatility	σ_y	0.250
Stock price volatility	σ_s	0.200
Correlation between stock and oil prices	ρ_{yS}	0.040
Risk aversion coefficient	γ	2.046
Discount rate	δ	0.030
Time to depletion	$\hat{T}$	70
Initial financial wealth income ratio	W_{2013}/y_{2013}	13
Initial financial wealth GDP ratio	W_{2013}/GDP_{2013}	2.500
GDP growth rate	$\log(GDP_t/GDP_{t-1})$	0.015

Notes: Calibrated parameters use Norwegian annual data for the period 1900–2013. Nominal prices are deflated by the U.S. CPI deflator from U.S. Bureau of Labor Statistics (2014). α_p is calibrated as the mean growth rate of real WTI oil price. WTI oil price is collected from Global Financial Data (2014). α_q is computed as the extraction rate at 2013 level in Norway. The production of oil is obtained from U.S. Energy Information Administration (2014). The crude oil reserve is obtained from BP (2014). α is computed following $\alpha = \alpha_p - \alpha_q$. The parameter values of r and ψ are from Campbell and Viceira (2002). σ_y and σ_s are the standard deviation of real oil price growth and real stock price growth. U.S. S&P 500 composite price index, the measure of stock price, is obtained from Global Financial Data (2014). ρ_{yS} is the correlation between the stock price growth and oil price growth. γ is computed by targeting the optimal $\pi_s = \frac{1}{\gamma}\frac{\psi}{\sigma_S^2} = 60\%$ at period $\hat{T}$. The value of δ follows Munk and Sørensen (2010). $\hat{T}$ is computed using Norwegian reserve of crude oil by assuming a constant extraction rate α_q. The W_{2013}/y_{2013} and W_{2013}/GDP_{2013} are computed using Norwegian financial wealth, oil income and GDP in 2013. The financial wealth is obtained from Norges Bank Investment Management (2013). The oil income is computed as the product of real oil price times extraction quantity, assuming zero production cost. The Norwegian GDP is collected from World Bank (2014). $\log(GDP_t/GDP_{t-1})$, a measure of expected GDP growth rate in Norway, follows the average growth of real mainland GDP in Norway for the period 2008–2015 reported by IMF (2014).

4.2 Incomplete markets

As Table 3 reports, the correlation coefficient between income growth and stock returns suggests that the risks associated with the former are not fully spanned by the latter. Therefore, markets are said to be incomplete since the fund's manager can not fully insure/hedge against income fluctuations in the financial markets. Under incomplete markets, the model introduced in Section 3 does not have a closed form solution and therefore we need to restore to numerical methods to compute an approximated solution. More specifically, we solve the maximized HJB equation which results after replacing the first order conditions (Equations 10 and 11) and the terminal condition in Equation (9) back into the dynamic programming Equation 8.

To simplify the numerical procedure we exploit the homogeneity of the value function, $J(W, y, t)$, as shown in Munk and Sørensen (2010), and reduce the dimensionality of the problem by rewriting the model in terms of two state variables as opposed to three. In particular, we redefine the value function in terms of only wealth-income ratio and time. Then, for a given terminal condition, we combine a backward iterative procedure in the time dimension starting at $\hat{T}$ with an implicit finite difference approximations in the wealth-income ratio dimension to solve the system of equations formed by the HJB equation at a set of grid points for the two state variables. Following Munk and Sørensen (2010), the first-order partial derivatives of the value function with respect to the state variables are discretized using an upwind differencing scheme, while the second-order partial derivatives are approximated using a second-order central difference method. The model is solved with and without short-sales constraints. Similar methods to solve continuous-time dynamic programming have been employed in Brennan et al. (1997), Candler (1998), Munk and Sørensen (2010) and Achdou et al. (2015). The details on the algorithm can be found in Appendix 5.

Unconstrained solution. We first illustrate the quantitative implications for optimal asset allocation for the case where the manager of a commodity SWF does not face any leverage constraints. Figure 4 plots the mean and 95 percentile interval across 10000 simulations of the model's solution with each of the simulations covering a period of 70 years, i.e. until complete depletion of the resource. Panel (a) displays the optimal investment strategy in equity. It suggests that the manager should initially invest 20 percent above of his financial wealth in stocks in the first year. This result is explained by the relative large amount of underground commodity wealth available at the beginning of the investment horizon that allows the manager to borrow money against it and invest it in the risky asset which provides a higher return over the risk-free alternative. However, as the reserves of the commodity are depleted, the manager should decrease his holdings of the risky asset gradually until he reaches the myopic investment rule about 30 years. Panel (b) depicts the hedging demand component on the investment strategy given by Equation (11). Initially, this term represents around 6 percent of the initial share, then approaches zero monotonically as oil revenues gradually decrease.

The results in Panels (a) and (b) suggest that since the fund's commodity revenues are uncertain and the manager has a long term investment horizon, the optimal strategy should be 6 percent lower than what recommended by the standard myopic investment rule for short term investors. The smaller share in equities insures the fund against negative future fluctuations in the fund's income. This strategy is implemented by the manager after reducing the amount of short-selling operations used to leverage his otherwise high exposition to risky assets.

Panel (c) indicates that the consumption to financial wealth ratio is relatively constant at 4 percent over the sample period, which is close to the current consumption/transfer policy followed by the Norwegian government. Finally, Panel (d) plots the financial wealth to GDP ratio by assuming that the non-oil GDP of Norway will grow at a constant rate of 1.5 percent per year during the next 70 years. The GDP growth ratio is computed as the average growth of real mainland GDP in Norway for the period 2008–2015 reported by IMF (2014). Therefore, the initial financial wealth-GDP ratio is set to its 2013 value of 2.5 and according to our model it will increase to 5.8 by the time the natural resource is fully depleted.

The effect of oil price volatility. Panel (a) in Figure 5 depicts the annual variation of the oil price from 1900 to 2013. The plot clearly indicates that the volatility of the oil price changes over time in long-run samples. In particular we notice two different regimes of volatility: a high variability regime during the periods 1900-1940 and 1973-2013 and a low volatility episode from 1942 to 1973. To understand the role played by the volatility of oil price growth we assess how the optimal investment strategy changes for different volatility scenarios. Recall that in our benchmark calibration we have simply taken the average during the entire sample period 1900-2014. Therefore, in panel (b) we plot the optimal investment share in risky assets for three different volatilities of the oil price while keeping the remaining parameters as in the benchmark calibration, in particular a correlation coefficient between oil price changes and stock returns of 4 percent: (i) $\sigma_y = 0.01$ representing the low volatility period in 1942-1973, (ii) $\sigma_y = 0.34$ for the high volatility period of 1974-2014, and (iii) $\sigma_y = 0.25$ which corresponds to the variability in the whole sample.

The results suggest that for low expected commodity income volatility, the fund's manager should take a higher position in risky assets relative to the case where the expected oil price volatility is high. This is due to the low demand for hedging resulting from reduced income

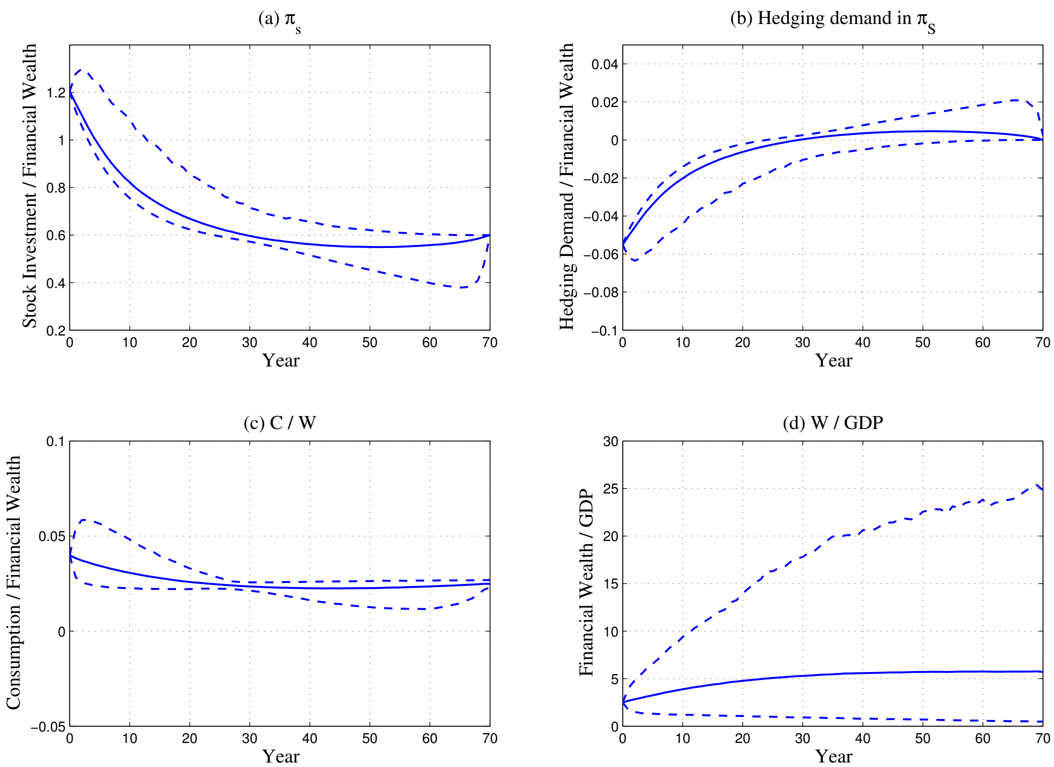

Figure 4: Optimal asset allocation under incomplete markets for Norway

Notes: Panels (a)–(d) plot the optimal investment strategy in equity, the hedging demand, the consumption rule as a fraction of financial wealth and the evolution of financial wealth to GDP ratio under incomplete markets for Norwegian economy following the calibration in Table (2). Solid lines represent the average value over 10000 simulated paths. Each of the paths contains 70 sample points. The dashed lines represent the 5 and 95 percentile from the empirical sampling distribution of simulated series.

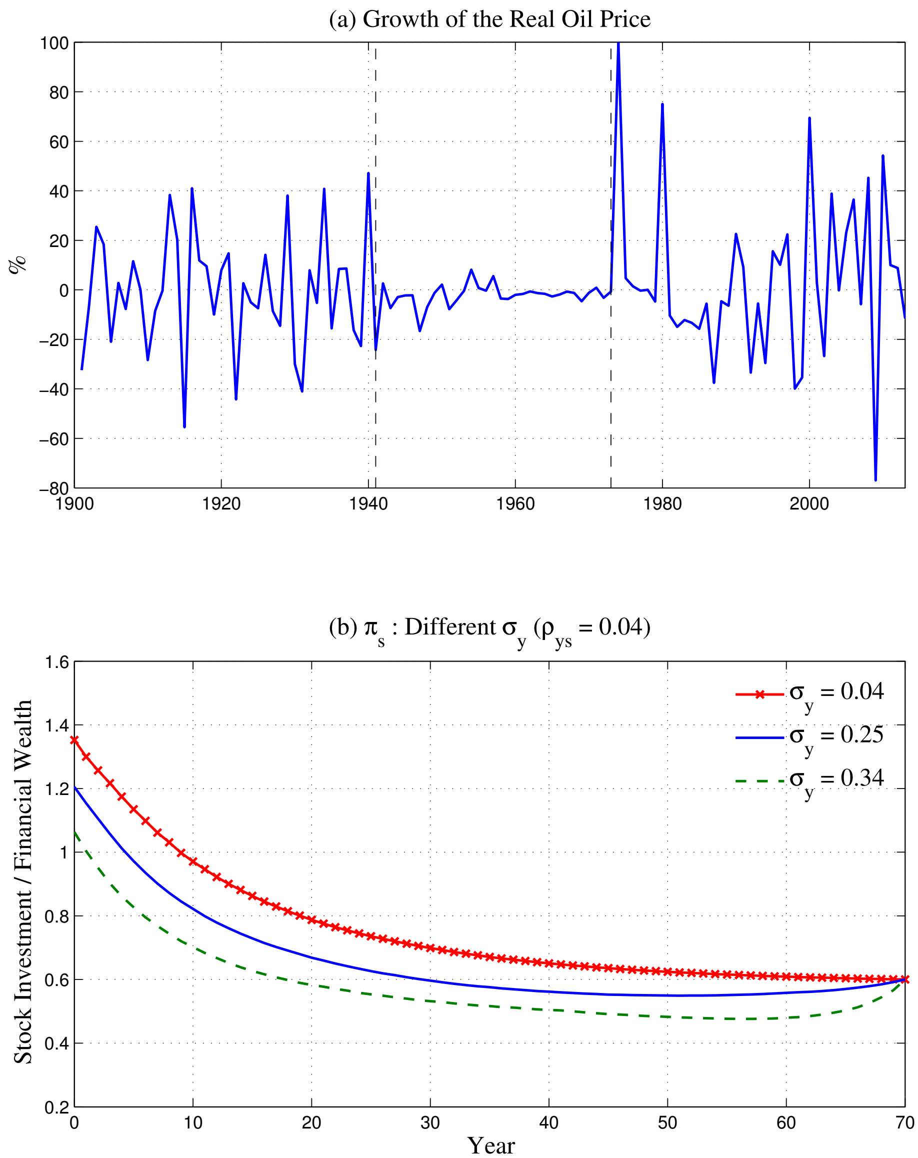

Figure 5: Optimal allocation for different stock-income correlations and oil volatilities

Notes: Panel (a) illustrates the annual growth of real oil price for the period 1900–2013. The vertical dash lines indicate the sub-periods for real oil price. The volatility of oil price growth is 0.04 for the period 1942–1973, 0.34 for the period 1974–2013 and 0.25 for the period. Panel (b) illustrates the sensitively of optimal investment on stock to the oil income volatility σ_y. The dash-dot curve is for $\sigma_y = 0.04$. The solid curve is for $\sigma_y = 0.25$. The dash curve is for $\sigma_y = 0.31$. For other parameters, the benchmark values from Table 2 are used. Panel (b) plots the mean value of 10,000 simulated paths.

risks. The opposite holds when the expected volatility of oil price is high. In this case, SWF managers should allocate less of the financial wealth in the stock market. In general, we find that the allocation decision is substantially affected by the volatility regime. From the perspective of time $t = 0$, the difference in equity allocations can be as high as 30 percent of the fund's financial wealth when moving from a low to high volatility state[6]. So far our experiment only considers the case of either low or high oil price volatility. Future research should allow for a time varying volatility of oil price changes such that the fund's manager includes his expectations about the future development of the volatility in his optimal asset allocation. At this point, we conjecture than the optimal allocation in Equation (11) will be extended to include a new hedge-demand component that will allow the manager to insure his portfolio returns against volatility risks.

Effect of constraints. As discussed previously, the optimal investment strategy dictates that the investor should hold an initial position in stocks above his initial financial wealth. In practice, however, many commodity SWF are liquidity constrained in that it may not be feasible to borrow money to achieve such a portfolio. To understand the effects of such type of constraints we compute the optimal investment when the SWF's ability to short-sale risk free assets to increase his position in equity is restricted according to $0 \leq \pi_{St} \leq 1$. Figure 6 plots the results. In general, we find a moderate effect of short-sale constraints in the optimal allocation of financial wealth in equity for the case of Norway. Indeed, as expected since Norway has an initial high wealth-income ratio the constraint is likely to be binding with low probability. The only effect is a moderate adjustment initially in the share of the risky asset. Also the consumption-wealth ratio has mild initial adjustment to reach higher consumption-wealth levels at the end of the period.

Loss due to suboptimal investments. Our optimal investment policy adjusts continuously the holdings in equities such that they reflect the economic conditions given by time variation in the state variables $(W_t, O(y_t, t), t)$. In reality, however, commodity SWFs are sometimes bounded and regulated to keep their investment policies within some ranges set through sovereign mandates. To shed some light into the potential losses of following a constant investment rule versus the optimal time-varying rule, we compute the real financial wealth gains of behaving optimally as predicted by our model. Panel (a) in Figure 7 plots the optimal and constant rule. The latter is fixed to the myopic strategy of 60 percent while the former replicates the investment rule in Figure 6. Panel (b) reports the gross gains in financial wealth from following the optimal rule versus the constant rule. It is clear from the

[6] The share invested in the risky asset is 1.35 for the low volatility scenario and 0.94 for the high volatility case.

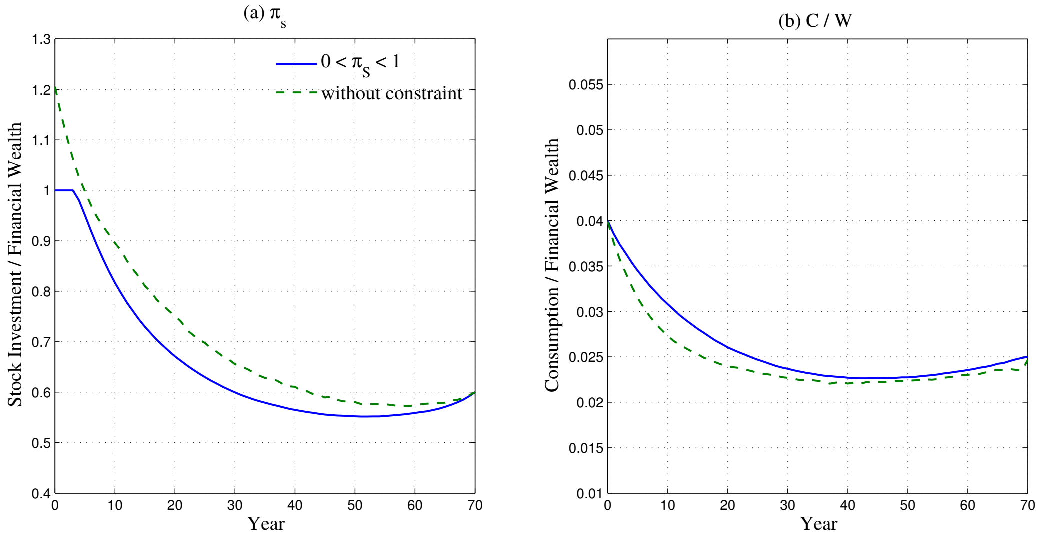

Figure 6: Optimal asset allocation with liquidity constraint

Notes: Panel (a) and (b) illustrates the sensitivity of the optimal investment on stocks and consumption to financial wealth ratio to liquidity constraints of the type $0 \leq \pi_{St} \leq 1$. The solid lines represent the average value over 10000 simulated paths of the optimal policy functions with liquidity constrains. Each of the paths contains 70 sample points. The dashed lines denote the case where the policy functions are unconstrained. The parameters used in the solution and simulation of the model are those in Table 2.

results, that real financial wealth is almost doubled close to the depletion time of the natural resource when following the time-varying asset allocation strategy, and is never below the wealth that can be achieved by following the myopic rule[7].

4.3 Comparison among countries

This section compares the optimal investment strategies among different SWFs. In particular, we contrast the dynamic behavior of the funds in Norway, UAE and Chile. Table 3 reports the calibration used for each the investment funds. The first column replicates the values used in the benchmark calibration in Table 2. For the SWF in the UAE (Abu Dhabi Investment Authority) we use the same commodity price and stock return parameters as for Norway, but different technical parameters associated with the extraction rate, initial wealth and income values. As shown in the second column, we fix the extraction rate of crude oil in the UAE at 4 percent per year which implies a negative drift for oil income of 3 percent per annum. The financial wealth is set at 7 times that of oil income in 2013. With these assumptions we estimate that complete depletion of the resource will be achieved in 170 years. In the third column we summarize the parameters used for the SWF in Chile (The Social and Economic Stabilization Fund in Chile) in which case the main funding source is that from copper

[7]To facilitate the comparison we use the optimal rule $c(W_t, y_t, t)$ in both cases. Future research should adjust the optimal consumption rule to the constant investement rule.

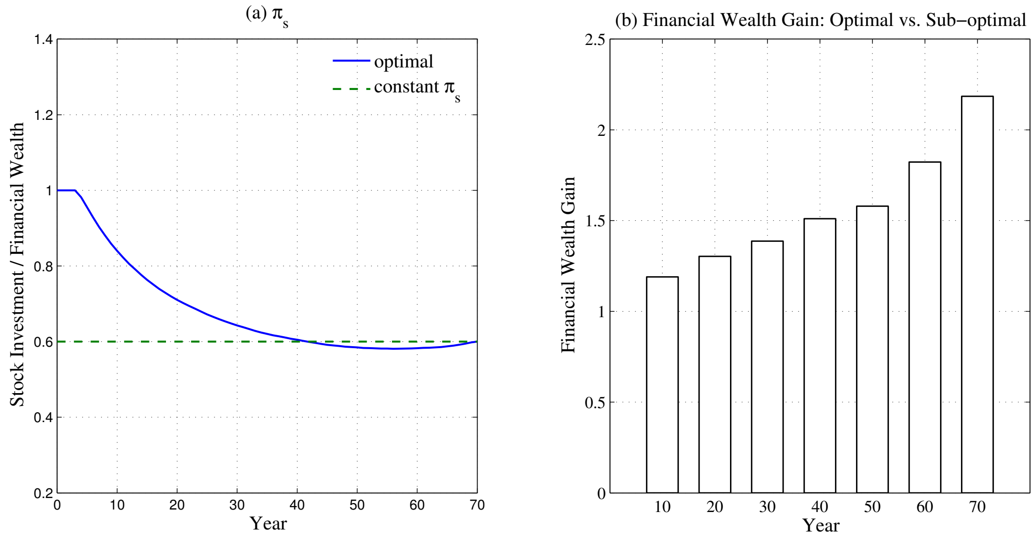

Figure 7: Comparison between optimal and constant investment share

Notes: Panels (a) and (b) compare the investment share and wealth-GDP ratio between the case with constant investment share on stock and the optimal investment strategies with liquidity constraints $0 \leq \pi_s \leq 1$. The solid curve is for the case with optimal investment strategies, as in Equation (11), and consumption rule, in Equation (10). The dash line is for the suboptimal solution assuming constant investment share on risky asset, $\pi_s = 60\%$. The simulation employs parameters as shown in Table 2. The graph plots the mean value of 10,000 simulated paths.

exports. The average growth of the copper price is -0.4 percent over the period 1900-2013 as reported in Table 1. The annual production/extraction rate is set to be 3 percent. Therefore the expected copper income growth is -3.4 percent. Due to the relatively low extraction rate, the depletion time of copper is estimated to be 150 years. The correlation between the copper price inflation and the stock returns is 0.32 for the period 1900-2013. As a young sovereign wealth fund (created in 2007) the cumulated financial wealth is low and as of 2013 it only accounts for 38 percent of the copper income.

Comparing the UAE and Norway. Figure 8 compares optimal investment strategy for fund managers of Norway and the UAE under short-sale constraints. Panel (a) plots the average investment share on equities over 10000 simulations of the model's solution, each of them with 70 sample points. The results suggest that, given the current calibration, the UAE holds a higher position in the risky asset over time. This can be explained by two interconnected factors. First, the time to depletion is further out in time which implies their underground oil wealth is higher from the perspective of today. Second, the initial financial wealth to income ratio is lower than the one reported for Norway leading to a higher demand for equities as can be seen from Equation 11. Furthermore, the hedging demand plotted in panel (b) combined with a higher expected growth rate in oil revenues over time indicate that the SWF in the UAE can invest a higher fraction of its financial wealth into stocks relative to

Table 3: Calibrated parameters for different countries (period 1900-2013)

Parameter	Norway	UAE	Chile
	Oil	Oil	Copper
α_p	0.010	0.010	-0.004
α_q	0.060	0.040	0.030
α	-0.050	-0.030	-0.034
r	0.006	0.006	0.006
ψ	0.050	0.050	0.050
σ_y	0.250	0.250	0.250
σ_s	0.200	0.200	0.200
ρ_{yS}	0.040	0.040	0.321
γ	2.046	2.000	2.000
δ	0.030	0.030	0.030
$\hat{T}$	70	170	150
W_{2013}/y_{2013}	13.00	7.00	0.38
W_{2013}/GDP_{2013}	2.5	3.0	0.1
$\log(GDP_t/GDP_{t-1})$	0.015	0.015	0.015

Notes: Calibrated parameters use annual data for the period 1900–2013. Nominal prices are deflated by the U.S. CPI deflator from U.S. Bureau of Labor Statistics (2014). α_p is calibrated as the mean growth rate of real WTI oil price and real high grade copper price. Commodity prices are collected from Global Financial Data (2014). α_q is computed as the extraction rate at 2013 level in Norway, UAE and Chile. The production of oil is obtained from U.S. Energy Information Administration (2014). The copper production in Chile is from Chilean Copper Commission (2014). The crude oil reserve is obtained from BP (2014). The Chilean reserve of copper is collected from U.S. Geological Survey (2014). α is computed following $\alpha = \alpha_p - \alpha_q$. The parameter values of r and ψ are from Campbell and Viceira (2002). σ_y and σ_s are the standard deviation of real commodity price growth and real stock price growth. U.S. S&P 500 composite price index, the measure of stock price, is obtained from Global Financial Data (2014). ρ_{yS} is the correlation between the stock price growth and commodity price growth. γ is computed by targeting the optimal $\pi_s = \frac{1}{\gamma}\frac{\psi}{\sigma_S^2} = 60\%$ at period $\hat{T}$. The value of δ follows Munk and Sørensen (2010). $\hat{T}$ is computed using reserve of crude oil and copper by assuming a constant extraction rate α_q for each country. The W_{2013}/y_{2013} and W_{2013}/GDP_{2013} are computed using financial wealth, oil income and GDP in 2013. The financial wealth in Norway, UAE and Chile are obtained from Norges Bank Investment Management (2013), SWF Institute (2014) and Ministry of Finance Gobierno de Chile (2013) respectively. The commodity income is computed as the product of real commodity price times extraction quantity, assuming zero production cost. The GDP of three countries are collected from World Development Indicators (2013). $log\,(GDP_t/GDP_{t-1})$, a measure of expected GDP growth rate in Norway, follows the average growth of real mainland GDP in Norway for the period 2008–2015 reported by IMF (2014).

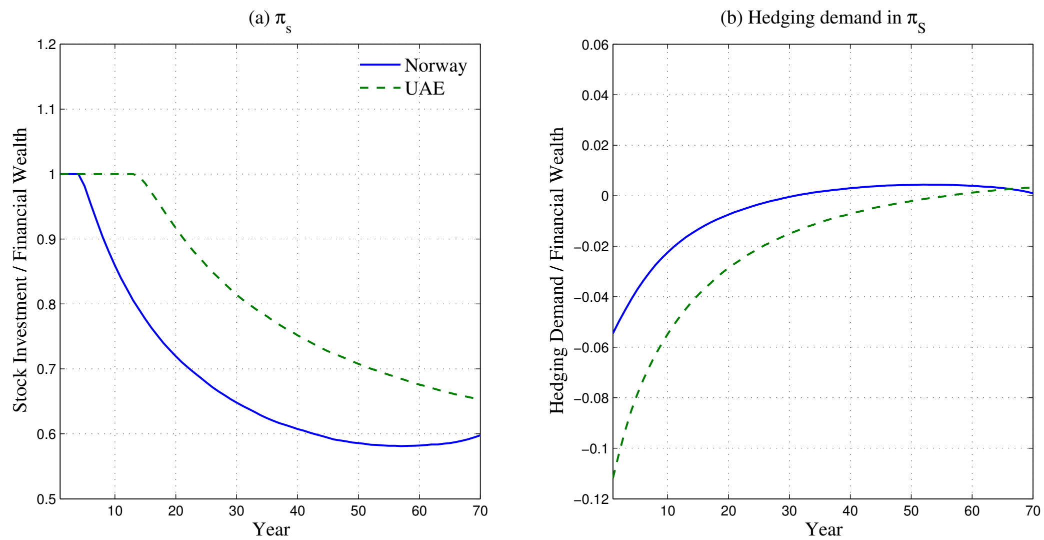

Figure 8: Optimal allocation under incomplete market for Norway and UAE:

Notes: Panels (a) and (b) plot optimal investment strategies and financial wealth GDP ratio for Norway, UAE and Chile. The solid line is for Norway, the solid-cross line is for UAE, and the dash line is for Chile. The simulations employ parameters as shown in Table 3 for each country. The liquidity constraints $0 \leq \pi_s \leq 1$ are imposed when solving the model numerically. The graph plots the mean value of 10,000 simulated paths.

Norway without compromising the fund's ability to smooth its transfers to the government. Ceteris paribus, this leads us to conclude that the optimal share of financial wealth invested in risky assets at every point in time is an increasing function of time to depletion, income to financial-wealth ratio and expected growth in commodity revenues.

The case of Chile. We would like to finish this section with some comments regarding the Chilean SWF where copper is the source of income. While the price of copper exhibits a similar level of volatility as the price of oil for all the sub-periods considered in this paper, its correlation with the stock returns is higher than the one reported for the price of oil. In particular, for the period 1900-2013, the sample correlation between stock returns and price changes in copper is about eight times larger than that for the oil price. Meanwhile, Chile may have higher risk aversion. Given the parameter values for Chile, Figure 9 plots the optimal investment strategy for different values of the coefficient of relative risk aversion. Interestingly, we find that for a coefficient of relative risk aversion $\gamma = 3$, the optimal allocation policy in the risky asset is reverted in the sense that now the manager should start investing all his financial wealth in risk free assets for the first 30 years (even short-sale equities if possible to leverage this strategy) and then increase gradually his position in equities to reach a steady state fraction of about 40 percent in the long run.

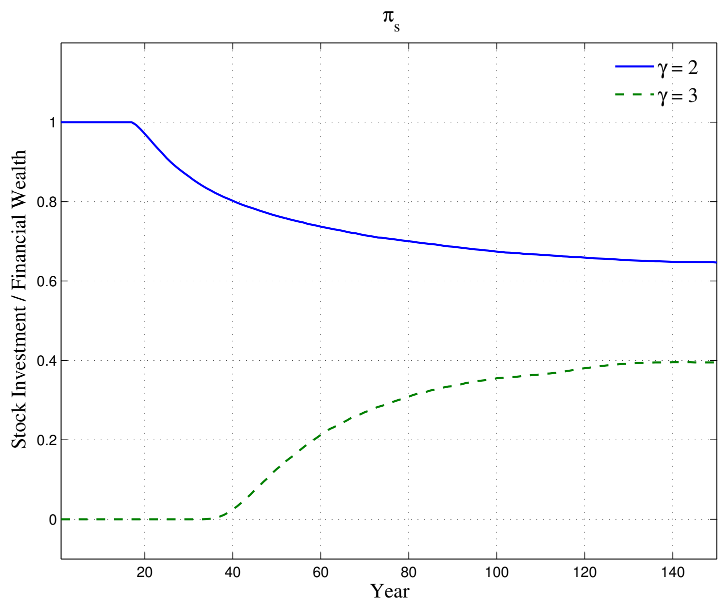

Figure 9: Optimal allocation with different relative risk aversion under incomplete market for Chile:

Notes: The figure plots optimal investment strategies on risky asset for Chile using different relative risk aversion parameter γ. The solid line is for $\gamma = 2$, the dash line is for $\gamma = 3$. The simulations employ other parameters as shown in Table 3 for Chile. The liquidity constraints $0 \leq \pi_s \leq 1$ are imposed when solving the model numerically. The graph plots the mean value of 10,000 simulated paths.

5 Conclusions

Sovereign wealth funds have become an important player in the global asset markets during the last 15 years. They represent 3 percent of the global investment industry and their asset holdings far exceed those of hedge funds. Therefore, this paper investigates the optimal portfolio allocation of a commodity SWF with a long-term investment horizon by setting up and solving an otherwise standard dynamic asset allocation problem under incomplete markets along the lines of Bodie et al., 1992 and Campbell and Viceira (2002). To bring the model closer to the economic environment faced by a SWF, we calibrate the fund's income process to replicate the volatility and correlation of commodity price changes and stock returns.

To gain analytical tractability and useful insights about the dynamics and mechanisms behind the model, we first solve the model under the assumption of complete market following Munk and Sørensen (2010). We find that when income risk is fully spanned by the stock market, the manager's optimal asset allocation depends mainly on the dynamics of the oil wealth to financial wealth ratio, where the former is defined as the net present value of all future revenues coming from commodity sales. We also find that the decision to increase or decrease the demand for risky assets depends crucially on the level of relative risk aversion, the volatility of commodity income and its correlations with the stock returns.

With this information at hand, we then proceed to solve the model under the assumption of incomplete markets in an attempt to bring the model closer to reality. Two types of experiments are carried out. We first compute the optimal behavior for unconstrained investment strategies and then we impose liquidity constraints of the short-sales type. The model is calibrated to resemble the conditions for the Norwegian SWF and simulated to quantitatively assess different investment policies.

For the unconstrained case we find that the SWF in Norway should initially allocate more than 100 percent of their financial wealth into risky assets and then gradually decrease its holdings of stocks to reach a long run share of 60 percent. This position should be achieved after 30 years. When the fund's manager is short-sale constrained, the optimal solution suggests to keep all the financial wealth invested in stocks for the first five years and then start decreasing gradually its holdings to the long run share of 60 percent.

In long-run samples, the growth in the price of oil exhibits periods of low and high volatilities. Therefore, we conduct a sensitivity analysis to assess the effect of different levels of commodity income volatility. We find that different volatility regimes can substantially affect the size of the hedging demand with differences of up to 30 percent in the share invested in stocks when going from a low to a high volatility episode.

We finally perform a similar analysis for two other commodity SWFs in the UAE and Chile. We find that given the higher underground oil wealth reserves than Norway, the UAE should follow an investment strategy in equities that far exceeds the optimal rule in Norway. This experiment leads us to conclude that, ceteris paribus, the optimal share of financial wealth invested in risky assets at every point in time is an increasing function of time to depletion, income to financial-wealth ratio and expected growth in commodity revenues. For the case of Chile, we find that for reasonable values of the relative risk aversion coefficient, it may be optimal for the fund's manager to start investing all his wealth in risk-free assets and then gradually decrease his position to create a portfolio that also includes risky assets.

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Appendix

A. Commodity wealth under complete markets

In this appendix we solve for the reduced form solution of oil wealth under complete market with $\rho_{yS} = \pm 1$.

When the income is spanned with stock price, the commodity income can be valued as the dividend stream from the traded stock

$$\frac{dy_t}{y_t} = \alpha dt + \sigma_y \rho_{yS} dz_{St}. \tag{18}$$

The risk-neutral income process can be written as:

$$\frac{dy_t}{y_t} = \left(\alpha - \sigma_y \rho_{yS} \frac{\psi}{\sigma_S}\right) dt + \sigma_y \rho_{ys} y dz_{St}^{\mathbb{Q}}. \tag{19}$$

where $z_{St}^{\mathbb{Q}}$ is the Wiener process under risk neural probability $\mathbb{Q}$, where

$$z_{St}^{\mathbb{Q}} = \frac{\psi}{\sigma_S} t + z_{st}.$$

According to the Feynman-Kac Theorem, the value of a future commodity income at time t takes the form:

$$O(y_t, t) = E_t^{\mathbb{Q}} \left[\int_t^{\hat{T}} e^{-r(s-t)} y_s ds \right] \tag{20}$$

given the boundary condition $O\left(y, \hat{T}\right) = 0$. To be noticed that the expectation is under the risk-neutral probably $\mathbb{Q}$.

According to (19), we may solve for y_s as:

$$\begin{aligned} y_s &= y_t \exp\left\{ \left(\alpha - \sigma_y \rho_{yS} \frac{\psi}{\sigma_S} - \frac{1}{2}\sigma_y^2\right)(s-t) + \int_t^s \sigma_y \rho_{yS} dz_{Su}^{\mathbb{Q}} \right\} \\ &= y_t \exp\left\{ \left(\alpha - \sigma_y \rho_{yS} \frac{\psi}{\sigma_S} - \frac{1}{2}\sigma_y^2\right)(s-t) + \sigma_y \rho_{yS} \left(z_{Ss}^{\mathbb{Q}} - z_{St}^{\mathbb{Q}}\right) \right\}, \end{aligned}$$

and then

$$e^{-r(s-t)} y_s = y_t \exp\left\{ \left(-r + \alpha - \sigma_y \rho_{yS} \frac{\psi}{\sigma_S} - \frac{1}{2}\sigma_y^2\right)(s-t) + \sigma_y \rho_{yS} \left(z_{Ss}^{\mathbb{Q}} - z_{St}^{\mathbb{Q}}\right) \right\}. \tag{21}$$

Known that $z_{Ss}^{\mathbb{Q}} - z_{St}^{\mathbb{Q}}$ is normally distributed, we may apply the standard rule for expectations of exponential of normal random variables on Equation (21). That is

$$\begin{aligned}
E_t^{\mathbb{Q}}\left[e^{-r(s-t)}y_s ds\right] &= y_t E_t^{\mathbb{Q}} \exp\left\{\left(-r+\alpha-\sigma_y\rho_{yS}\frac{\psi}{\sigma_S}-\frac{1}{2}\sigma_y^2\right)(s-t)+\sigma_y\rho_{yS}\left(z_{Ss}^{\mathbb{Q}}-z_{St}^{\mathbb{Q}}\right)\right\} \\
&= y_t e^{F(t,s)-r+\alpha-\sigma_y\rho_{yS}\frac{\psi}{\sigma_S}-\frac{1}{2}\sigma_y^2} \qquad (22)
\end{aligned}$$

where $F(t,s)$ is some function can be computed easily. Integrating (22) over s we obtain equation (12).

B. Closed form solution under complete market

This appendix solves for the closed form solution under complete market with $\rho_{yS} = \pm 1$.

The social planner faces the problem of maximizing the expected present value of utility such that

$$J(W_t, y_t, t) = \max_{\{c_u, \theta_{Su}\}_{u=t}^T} \mathbb{E}_t\left[\int_t^T e^{-\delta t}U(c_u)\,du + \epsilon e^{-\delta(T-t)}U(W_T)\right].$$

We assume CRRA utility function: $U(c_t) = \frac{1}{1-\gamma}c_t^{1-\gamma}$.

We assume the investor has access to two tradable assets. The bond is the risk-free asset whose price evolves following:

$$\frac{dB_t}{B_t} = rdt. \qquad (23)$$

The stock is the risky asset, whose price evolves as the process:

$$\frac{dS_t}{S_t} = (r+\psi)\,dt + \sigma_S dz_{St}. \qquad (24)$$

Under complete market, the income rate is spanned by the risky asset. The income evolves following:

$$dy_t = y_t\left[\alpha dt + \sigma_y\rho_{yS}dz_{St}\right]$$

where $\rho_{yS} = \pm 1$.

The dynamic of wealth evolves following:

$$dW_t = \theta_S\frac{dS_t}{S_t} + (y_t - c_t)\,dt + (W_t - \theta_{St})\,rdt. \qquad (25)$$

Inserting (24) into (25), we obtain that:

$$\begin{aligned}
dW_t &= \left[(r+\psi)\theta_{St} + r(W_t - \theta_{St}) + y_t - c_t\right]dt + \sigma_S\theta_S dz_{St} \\
&= \left[rW_t + \psi\theta_{St} + y_t - c_t\right]dt + \sigma_S\theta_S dz_{St}.
\end{aligned}$$

The terminal condition of the value function $J(W,y,t)$ is given such that:

$$J(W,y,T)=\epsilon U(W_T)=\frac{\epsilon W_T^{1-\gamma}}{1-\gamma} \tag{26}$$

where W_T is the terminal level of wealth.

Then the HJB function under complete market is given by:

$$\begin{aligned}\delta J(W_t,y_t,t) &= \max_{c_t,\theta_{St}} U(c_t)+J_t(W_t,y_t,t)+J_W\left[rW_t+\psi\theta_{St}+y_t-c_t\right]\\ &\quad +\frac{1}{2}J_{WW}(W_t,y_t,t)\,\sigma_S^2\theta_{St}^2\\ &\quad +\alpha yJ_y(W_t,y_t,t)+\frac{1}{2}J_{yy}(W_t,y_t,t)\,y^2\sigma_y^2\\ &\quad +J_{Wy}(W_t,y_t,t)\,y_t\theta_{St}\sigma_S\sigma_y\rho_{yS} \end{aligned}\tag{27}$$

where J_i denotes the first order partial derivative the value function with respect to the state variable i, and J_{ij} the second order derivative with respect to the state variables i and j.

The first order conditions gives solution for θ_S and c as follow:

$$\theta_{St} = -\frac{J_W}{J_{WW}}\frac{\psi}{\sigma_S^2}-\frac{y_tJ_{Wy}}{J_{WW}}\frac{\sigma_y\rho_{yS}}{\sigma_S} \tag{28}$$

$$c_t = J_W^{-\frac{1}{\gamma}}. \tag{29}$$

While looking for analytical solution of value function $J(W_t,y_t,t)$, we first conjecture the value function takes the form:

$$J(W_t,y_t,t)=\frac{1}{1-\gamma}g(t)^{\gamma}\left(W_t+O(y_t,t)\right)^{1-\gamma} \tag{30}$$

with unknown function form of $g(t)$. $O(y_t,t)$ is given in the complete market case which follows

$$\begin{aligned}O(y_t,t) &= \begin{cases}\frac{y_t}{r-\alpha+\sigma_y\rho_{yS}\frac{\psi}{\sigma_S}}\left(1-e^{-\left(r-\alpha+\xi\frac{\psi}{\sigma_S}\right)\left(\hat{T}-t\right)}\right) & \text{if } r-\alpha+\sigma_y\rho_{yS}\frac{\psi}{\sigma_S}\neq 0\\ y_t\left(\hat{T}-t\right) & \text{if } r-\alpha+\sigma_y\rho_{yS}\frac{\psi}{\sigma_S}=0\end{cases}\\ &= y_tM(t).\end{aligned}$$

According to (30), we may get the derivatives of value function

$$\begin{aligned} J_t &= \frac{\gamma}{1-\gamma} g^{\gamma-1}(W+O)^{1-\gamma} g_t + g^{\gamma}(W+O)^{-\gamma} O_t \\ J_W &= g^{\gamma}(W+O)^{-\gamma} \\ J_y &= g^{\gamma}(W+O)^{-\gamma} O_y \\ J_{WW} &= -\gamma g^{\gamma}(W+O)^{-\gamma-1} \\ J_{yy} &= g^{\gamma}\left[(W+O)^{-\gamma} O_{yy} - \gamma (W+O)^{-\gamma-1}(O_y)^2\right] \\ J_{Wy} &= -\gamma g^{\gamma}(W+O)^{-\gamma-1} O_y. \end{aligned} \tag{31}$$

We drop the time subscript t and functional notations for simplicity, for example $O = O(y_t, t)$ and $g = g(t)$. Moreover, J_t denotes the first order partial derivative the value function with respect to time t. Inserting into the first order conditions, we get

$$\begin{aligned} \theta_S &= \frac{(W+O)}{\gamma}\frac{\psi}{\sigma_S^2} + yO_y \frac{\sigma_y \sigma_S \rho_{yS}}{\sigma_S^2} \\ &= \frac{1}{\sigma_S^2}\left\{\frac{(W+O)}{\gamma}\psi + yO_y \sigma_y \sigma_S \rho_{yS}\right\} \end{aligned} \tag{32}$$

and

$$c = \frac{W+O}{g(t)}. \tag{33}$$

Inserting (31), (32) and (33) into (27), and knowing that $O_y = M$, we get

$$\begin{aligned} \delta \frac{1}{1-\gamma} g^{\gamma}(W+O)^{1-\gamma} &= \frac{\gamma}{1-\gamma}\left(g^{\gamma-1}(W+O)^{1-\gamma}\right) + \frac{\gamma}{1-\gamma} g^{\gamma-1}(W+O)^{1-\gamma} g_t + g^{\gamma}(W+O)^{-\gamma} O_t \\ &\quad + g^{\gamma}(W+O)^{-\gamma}\left[rW + \frac{1}{\sigma_S^2}\left[\frac{(W+O)}{\gamma}\psi^2 + yM\sigma_y(\sigma_S \rho_{yS})\psi\right] - g^{-1}(W+O) + y\right] \\ &\quad - \frac{1}{2}\gamma g^{\gamma}(W+O)^{-\gamma-1}\left[\frac{1}{\sigma_S^2}\left\{\frac{(W+O)}{\gamma}\psi + yM\sigma_y\sigma_S\rho_{yS}\right\}^2\right] \\ &\quad + \alpha y g^{\gamma}(W+O)^{-\gamma} O_y + \frac{1}{2} g^{\gamma}\left[(W+O)^{-\gamma} O_{yy} - \gamma(W+O)^{-\gamma-1}(O_y)^2\right] y^2 \sigma_y^2 \rho_{yS}^2 \\ &\quad - \gamma g^{\gamma}(W+O)^{-\gamma-1} O \theta_S \sigma_y \sigma_S \rho_{yS}. \end{aligned}$$

Rewriting the term $rW = r(W+O) - rO$ and factorizing terms with $g^{\gamma}(W+O)^{1-\gamma}$ and $g^{\gamma}(W+O)^{-\gamma}$, we obtain

$$\begin{aligned} 0 &= g^{\gamma}(W+O)^{1-\gamma}\left\{-\delta\frac{1}{1-\gamma} + \frac{\gamma}{1-\gamma} g^{-1} + \frac{\gamma}{1-\gamma} g^{-1} g_t + r + \frac{1}{2}\frac{1}{\gamma}\frac{\psi^2}{\sigma_S^2}\right\} \\ &\quad + g^{\gamma}(W+O)^{-\gamma}\left\{O_t - rO + y + \left(\alpha - \sigma_y \rho_{yS}\frac{\psi}{\sigma_S}\right) yO_y + \frac{1}{2} O_{yy} y^2 \sigma_y^2 \rho_{yS}^2\right\}. \end{aligned} \tag{34}$$

Applying Feynman-Kac theorem, with given the solution of O under complete market in Equation

(20) and risk-neutral process of y in (19), we obtain the PDE in $O(y,t)$ as follow

$$O_t + \left[\alpha - \sigma_y \rho_{yS} \frac{\psi}{\sigma_S}\right] y O_y + \frac{1}{2}\sigma_y^2 \rho_{yS}^2 y^2 O_{yy} - rO + y = 0. \tag{35}$$

Therefore the second term in the (34) is zero. The (34) turns to

$$g_t = \frac{1}{\gamma}\left[\delta - r(1-\gamma) - \frac{1-\gamma}{2\gamma}\frac{\psi^2}{\sigma_S^2}\right] g - 1$$

Then we can solve for the function $g(t)$ give as:

$$g(t) = \frac{1}{A}\left[1 + (BA - 1)\, e^{-A(T-t)}\right]$$

where

$$A = \frac{1}{\gamma}\left[\delta - r(1-\gamma) - \frac{1-\gamma}{2\gamma^2}\|\lambda\|^2\right]$$

and B is a unknown constant. According to the terminal condition (26), we get

$$B = g(T) = \epsilon^{\frac{1}{\gamma}}.$$

Therefore

$$g(t) = \frac{1}{A}\left[1 + \left(\epsilon^{\frac{1}{\gamma}} A - 1\right) e^{-A(T-t)}\right] \tag{36}$$

or

$$g(t) = \frac{1}{A}\left[1 - e^{-A(T-t)}\right] + \epsilon^{\frac{1}{\gamma}} e^{-A(T-t)}.$$

Inserting into (30), we obtain the solution of value function.

$$J(W,y,t) = \frac{1}{1-\gamma} g(t)^{\gamma} \left(W + O(y,t)\right)^{1-\gamma}$$

where

$$g(t) = \frac{1}{A}\left[1 - e^{-A(T-t)}\right] + \epsilon^{\frac{1}{\gamma}} e^{-A(T-t)}$$
$$A = \frac{1}{\gamma}\left[\delta - r(1-\gamma) - \frac{1-\gamma}{2\gamma^2}\lambda_S^2\right].$$

Thus the closed-form solutions for consumption and investment are

$$c = \frac{W + O(y,t)}{g(t)} \tag{37}$$

$$\theta_S = \frac{1}{\gamma}\left(W + O(y,t)\right)\frac{\psi}{\sigma_S^2} - O\frac{\sigma_y \rho_{yS}}{\sigma_S} \tag{38}$$

for $t \in [0,T]$. Dividing both (37) and (38) by W, we arrive at (16) and (17).

C. Numerical solution method

This appendix explains the algorithm of the numerical solution for the asset allocation model under incomplete market in Section 4.2.

C.1 The original problem

We are looking for solution of $J(W, y, t)$ that solves the Bellman Equation (8) which is a non-linear partial difference equation shown as

$$\begin{aligned} \delta J &= \frac{c^{1-\gamma}}{1-\gamma} + J_t + J_W \left[rW + \theta_S \sigma_S \lambda_S - c + y \right] \\ &\quad + \frac{1}{2} J_{WW} \theta_S^2 \sigma_S^2 \\ &\quad + J_y \alpha y + \frac{1}{2} J_{yy} y^2 \sigma_y^2 \\ &\quad + J_{Wy} y \theta_S \sigma_S \sigma_y \rho_{yS} \end{aligned} \tag{39}$$

where J_i denotes the first order partial derivative the value function with respect to the state variable i, and J_{ij} the second order derivative with respect to the state variables i and j.

According to the first order conditions, θ_s takes the form that

$$\theta_S = -\frac{J_W}{J_{WW}} \frac{\lambda_S}{\sigma_S} - \frac{y J_{Wy}}{J_{WW}} \frac{\sigma_y \rho_{yS}}{\sigma_S}$$

and

$$c = J_W^{-\frac{1}{\gamma}}.$$

The terminal condition at time T is known as

$$J(W, y, T) = \epsilon U(W) = \frac{\epsilon W^{1-\gamma}}{1-\gamma}.$$

We know that after depletion time $\hat{T}$ the oil income is zero. There exists an analytical solution for the asset allocation model without endowment, that is

$$J(W, y, t) = \frac{1}{1-\gamma} g(t)^{\gamma} W^{1-\gamma}, \text{ for } \hat{T} \leq t \leq T$$

where

$$\begin{aligned} g(t) &= \frac{1}{A} \left[1 - e^{-A(T-t)} \right] + \epsilon^{\frac{1}{\gamma}} e^{-A(T-t)} \\ A &= \frac{1}{\gamma} \left[\delta - r(1-\gamma) - \frac{1-\gamma}{2\gamma} \frac{\psi^2}{\sigma_S^2} \right]. \end{aligned}$$

Thus at depletion time $\hat{T}$, we have

$$J\left(W, y, \hat{T}\right) = \frac{1}{1-\gamma} g\left(\hat{T}\right)^{\gamma} W^{1-\gamma} \tag{40}$$

where

$$g\left(\hat{T}\right) = \frac{1}{A}\left[1 - e^{-A\left(T-\hat{T}\right)}\right] + \epsilon e^{-A\left(T-\hat{T}\right)}. \tag{41}$$

In this equation δ,γ, r, ρ_{yS}, α, σ_s, σ_y, ψ, ϵ and $\hat{T}$ are parameters.

C.2 The transformed problem

To drop one state variable we use some properties of the function J. Knowing that the function J is homogeneous of degree $1-\gamma$ in W and y (See Munk and Sørensen (2010)), i.e.

$$J\left(k\left(t\right)W, k\left(t\right)y, t\right) = k\left(t\right)^{1-\gamma} J\left(W, y, t\right) \Longrightarrow J\left(W, y, t\right) = k\left(t\right)^{\gamma-1} J\left(k\left(t\right)W, k\left(t\right)y, t\right).$$

Using it with $k\left(t\right) = e^{-\beta t}/y$, we obtain

$$J\left(W, y, t\right) = y^{1-\gamma} e^{-\beta(\gamma-1)t} J\left(x, e^{-\beta t}, t\right) \equiv y^{1-\gamma} F\left(x, t\right) \tag{42}$$

where we define $x = e^{-\beta t}W/y$.

Employing the relationship in (42) and $x = e^{-\beta t}W/y$, we obtained the relationship as follow:

$$\begin{aligned} \frac{\partial x}{\partial y} &= -e^{-\beta t}\frac{W}{y^2} = -\frac{x}{y} \\ \frac{\partial x}{\partial t} &= -\beta x \end{aligned}$$

$$\frac{\partial x}{\partial W} = \frac{e^{-\beta t}}{y}$$

$$
\begin{aligned}
J_t &= y^{1-\gamma}\left(F_t - \beta F_x x\right) \\
J_W &= y^{1-\gamma} F_x \frac{\partial x}{\partial W} = e^{-\beta t} y^{-\gamma} F_x \\
J_{WW} &= e^{-\beta t} y^{-\gamma} F_{xx} \frac{\partial x}{\partial W} \\
&= e^{-2\beta t} y^{-\gamma-1} F_{xx} \\
J_y &= (1-\gamma)\, y^{-\gamma} F + y^{1-\gamma} F_x \frac{\partial x}{\partial y} = (1-\gamma)\, y^{-\gamma} F - y^{1-\gamma} F_x \frac{x}{y} \\
&= y^{-\gamma}\left[(1-\gamma)\, F - F_x x\right] \\
J_{yy} &= y^{-\gamma-1}\left[-\gamma\,(1-\gamma)\, F + 2\gamma F_x x + F_{xx} x^2\right] \\
J_{Wy} &= e^{-\beta t}\left(-\gamma y^{-\gamma-1} F_x + y^{-\gamma} F_{xx} \frac{\partial x}{\partial y}\right) \\
&= e^{-\beta t}\left(-\gamma y^{-\gamma-1} F_x - y^{-\gamma-1} F_{xx} x\right) \\
&= e^{-\beta t} y^{-\gamma-1}\left(-\gamma F_x - F_{xx} x\right).
\end{aligned}
$$

Inserting into (39) and dividing both side of the equation by $y^{1-\gamma}$, we get

$$
\begin{aligned}
\delta F \;=\; & \frac{\left(\frac{c}{y}\right)^{1-\gamma}}{1-\gamma} + F_t - \beta F_x x + \left[r \frac{W}{y} e^{-\beta t} + \pi_S \frac{W}{y} e^{-\beta t} \sigma_S \lambda_S - \frac{c}{y} e^{-\beta t} + e^{-\beta t}\right] F_x \\
& + \frac{1}{2} \pi_S^2 \sigma_S^2 \underbrace{\left(\frac{W}{y} e^{-\beta t}\right)^2}_{x^2} F_{xx} \\
& + \alpha\left[(1-\gamma)\, F - F_x x\right] \\
& + \frac{1}{2} \sigma_y^2\left[-\gamma\,(1-\gamma)\, F + 2\gamma F_x x + F_{xx} x^2\right] \\
& + \sigma_y \pi_S \sigma_S \rho_{yP} e^{-\beta t} \frac{W}{y}\left(-\gamma F_x - F_{xx} x\right).
\end{aligned} \tag{43}
$$

Define $\hat{c} \equiv \frac{c}{y}$ and $\pi_s \equiv \frac{\theta_S}{W}$. Factoring terms with F in the Equation (43), then we are aiming to solve for $F\,(x,t)$ which solves the following PDE.

$$
\begin{aligned}
\hat{\delta} F \;=\; & \frac{\hat{c}^{1-\gamma}}{1-\gamma} + F_t \\
& + \left\{(1-\hat{c})\, e^{-\beta t} + x\left[r - \alpha + \pi_S \sigma_S\left(\frac{\psi}{\sigma_S} - \sigma_y \rho_{yP} \gamma\right) + \sigma_y^2 \gamma - \beta\right]\right\} F_x \\
& + \frac{1}{2}\left[\pi_S^2 \sigma_S^2 + \sigma_y^2 - 2\sigma_y \sigma_S \rho_{yS} \pi_S\right] x^2 F_{xx}
\end{aligned} \tag{44}
$$

where

$$
\hat{c} = F_x^{-\frac{1}{\gamma}} e^{\frac{\beta t}{\gamma}} \tag{45}
$$

and

$$\pi_S = -\frac{F_x}{xF_{xx}} \frac{\frac{\psi}{\sigma_S} - \gamma\sigma_y\rho_{yS}}{\sigma_S} + \frac{\sigma_y\rho_{yS}}{\sigma_S}. \tag{46}$$

In Equation (44), β, γ, r, ρ, σ and α are parameters, and $\hat{\delta}$ is

$$\hat{\delta} = \delta + \alpha(\gamma - 1) - \frac{1}{2}\sigma_y^2\gamma(\gamma - 1).$$

Thus we transform the original two-state PDE (39) into a problem solving for $F = F(x, t)$ with one state variable as shown in (44).

When solving Equation (44) numerically, we also impose a constraint that

$$0 \leq \pi_{St} \leq 1. \tag{47}$$

C.3 Solution algorithm

Define the state space. Before solving the PDE Equation (44) using finite difference methods, we set up an equally spaced lattice in (x, t) defined by the grid points

$$\{(x_i, t_n) \mid i = 0, 1, ..., I;\ n = 0, 1, ..., N\}$$

where $x_i = x_0 + i\triangle x$ and $t_n = n\triangle t$. We define initial values $x_0 = 0.1$, $x_I = 60$, and $N = 24 \times \hat{T}$ with $\hat{T} = 70$.

Terminal condition at $\hat{T}$. Given the condition at depletion time $\hat{T}$ in Equation (40), we may transform it into the condition for $F(x_i, t_N)$ at each value of x_i at time t_N ($t_N = \hat{T}$), which is

$$F_{i,N} = \frac{1}{1-\gamma} e^{-\beta(\gamma-1)\hat{T}} g\left(\hat{T}\right)^{\gamma} x_i^{1-\gamma} \tag{48}$$

where the function $g(\hat{T})$ follows Equation (41).

Terminal value of $\hat{c}_{i,N}$ is computed following (45). The terminal investment share is computed using (46)

$$\pi_S(x_i, t_N) = -\frac{F_x(x_i, t_N)}{xF_{xx}(x_i, t_N)} \frac{1}{\sigma_S}\left(\frac{\psi}{\sigma_S} - \gamma\sigma_y\rho_{yS}\right) + \frac{\sigma_y\rho_{yS}}{\sigma_S}.$$

While computing for control variables at time t_N, we use central difference method to approximate first and second derivatives of function F

$$\begin{aligned}F_x(x_i,t_N) &= \frac{F_{i+1,N}-F_{i-1,N}}{2\triangle x}\\F_{xx}(x_i,t_N) &= \frac{F_{i+1,N}-2F_{i,N}+F_{i-1,N}}{(\triangle x)^2}.\end{aligned}$$

Meanwhile, we employ forward difference and backward difference methods to approximate F_x and F_{xx} at the lowest and highest value of space variable x, $i=1$ and I, respectively, such as

$$F_x(x_0,t_N) = \frac{F_{1,N}-F_{0,N}}{\triangle x} \tag{49}$$

$$F_x(x_I,t_N) = \frac{F_{I,N}-F_{I-1,N}}{\triangle x} \tag{50}$$

$$F_{xx}(x_0,t_N) = \frac{F_{1,N}-2F_{0,N}+0}{(\triangle x)^2} \tag{51}$$

$$F_{xx}(x_I,t_N) = \frac{0-2F_{I,N}+F_{I-1,N}}{(\triangle x)^2} \tag{52}$$

at time t_N.

Iterations. There are two steps to solve the Equation (44). First, we use the implicit finite different method to solves (44) and iterate $F(x_i,t_n)$ backward from time $\hat{T}$ to 0 with given terminal condition (48) using a guess on the optimal controls $\pi_s(x_i,t_n)$ and $\hat{c}(x_i,t_n)$. Second, we employ the solution of $F(x_i,t_n)$ at time t_n from the first step and compute a new guess of the optimal controls which can again generate another solution of $F(x_i,t_n)$. The procedure carries on until the value of $F(x_i,t_n)$ gets convergence.

Backward iteration over time with a guess of control variables. With given terminal solution (48), we are going to iterate backward starting from $\hat{T}$. At any time t_n, we know the approximated value function $F_{i,n+1}$, and optimal control variables $\hat{c}_{i,n+1}$ and $\pi_{i,n+1}$ at time t_{n+1}. We start with a guess that the value of optimal control at time t_n is $\hat{c}_{i,n}=\hat{c}_{i,n+1}$ and $\pi_{i,n}=\pi_{i,n+1}$.

Defining that

$$\begin{aligned}a_{i,n} &= (1-\hat{c}_{i,n})e^{-\beta t}+x_{i,n}\left[r-\alpha+\sigma_y^2\gamma-\beta++(\psi-\gamma\rho_{yS}\sigma_y\sigma_S)\pi_{S;i,n}\right]\\B_{i,n} &= \frac{1}{2}\left[\sigma_S^2\pi_{S;i,n}^2+\sigma_y^2\right]x_{i,n}^2,\end{aligned}$$

we may write (44) as

$$\hat{\delta}F_{i,n} = \frac{\hat{c}_{i,n}^{1-\gamma}}{1-\gamma}+F_t(x_i,t_n)+a_{i,n}F_x(x_i,t_n)+B_{i,n}F_{xx}(x_i,t_n). \tag{53}$$

We rewrite Equation (53) using up-wind approximations of derivatives, and get

$$\hat{\delta} F_{i,n} = \frac{\hat{c}_{i,n}^{1-\gamma}}{1-\gamma} + D_t^+ F_{i,n} + D_x^+ F_{i,n} a^+ - D_x^- F_{i,n} a^- + D_x^2 F_{i,n} B \tag{54}$$

where $z^+ = \max\{z, 0\}$ and $z^- = \min\{z, 0\}$. $D_t^+ F$, $D_x^+ F$, $D_x^- F$ and $D_x^2 F$ represents

$$D_t^+ F_{i,n} = \frac{F_{i,n+1} - F_{i.n}}{\triangle t} \tag{55}$$

$$D_x^2 F_{i,n} = \frac{F_{i+1,n} - 2F_{i,n} + F_{i-1,n}}{(\triangle x)^2} \tag{56}$$

$$D_x^+ F_{i,n} = \frac{F_{i+1,n} - F_{i,n}}{\triangle x} \tag{57}$$

$$D_x^- F_{i,n} = \frac{F_{i,n} - F_{i-1,n}}{\triangle x}. \tag{58}$$

Inserting up-wind approximations of derivatives (55)–(58) into (54), we obtain

$$\begin{aligned} \hat{\delta} F_{i,n} = {} & \frac{\hat{c}_{i,n}^{1-\gamma}}{1-\gamma} + \frac{F_{i,n+1} - F_{i,n}}{\triangle t} \\ & + \frac{F_{i+1,n} - F_{i,n}}{\triangle x} a^+ - \frac{F_{i,n} - F_{i-1,n}}{\triangle x} a^- \\ & + \frac{F_{i+1,n} - 2F_{i,n} + F_{i-1,n}}{(\triangle x)^2} B. \end{aligned} \tag{59}$$

Moving variables with $n+1$ subscripts and $\hat{c}_n$ to the LHS and those with n subscripts to the RHS in Equation (59). I also factorize terms on the RHS. We got

$$\begin{aligned} \underbrace{-\frac{\hat{c}_{i,n}^{1-\gamma}}{1-\gamma} - \frac{F_{i,n+1}}{\triangle t}}_{d_{i,n}} = {} & F_{i,n} \underbrace{\left\{ \begin{array}{c} -\hat{\delta} - \frac{1}{\triangle t} - \frac{1}{\triangle r}\left(a_{i,n}^+ + a_{i,n}^-\right) - \frac{1}{\triangle x}\left(a_{i,n}^+ + a_{i,n}^-\right) \\ -\frac{2}{(\triangle x)^2} B_{i,n} \end{array} \right\}}_{E_{i,n}} \\ & + F_{i+1,n} \underbrace{\left\{ \frac{1}{\triangle x} a_{i,n}^+ + \frac{1}{(\triangle x)^2} B_{i,n} \right\}}_{H_{i,n}} \\ & + F_{i-1,n} \underbrace{\left\{ \frac{1}{\triangle x} a_{i,n}^- + \frac{1}{(\triangle x)^2} B_{i,n} \right\}}_{I_{i,n}} \end{aligned}$$

or

$$d_{i,n} = F_{i-1,n} I_{i,n} + F_{i,n} E_{i,n} + F_{i+1,n} H_{i,n}. \tag{60}$$

Writing Equation (60) into vectors and matrixes. At one specific time t_n, we have

$$\mathbf{d_n} = \mathbf{X_n F_n}$$

where $\mathbf{X_n}$ is a matrix with dimension $(I+1)\times(I+1)$. $\mathbf{F_n}$ is a vector with dimension of $(I+1)\times 1$. In the matrixes below, parameters are drop for the time subscript n, for example $E_i = E_{i,n}$.

$$\mathbf{X_n} = \begin{bmatrix} E_0 & H_0 & 0 & \cdots & 0 \\ I_1 & E_1 & H_1 & 0 & \vdots \\ 0 & I_2 & E_2 & H_2 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & 0 & I_I & E_I \end{bmatrix}$$

and

$$\mathbf{F_n} = \begin{bmatrix} F_0 \\ F_1 \\ \vdots \\ F_I \end{bmatrix}_{(I+1)\times 1}$$

At time t_n, $\mathbf{d_n}$ and $\mathbf{X_n}$ are known. Thus we can solve for unknown $\mathbf{F_n}$ as

$$\mathbf{F_n} = inv\left(\mathbf{X_n}\right)\mathbf{d_n}.$$

Iteration for value function at time t_n. Given the solution of $\mathbf{F_n}$ from the first step, we can compute approximated derivatives using central difference method. For $i = 2, ..., I-1$, we compute the derivative of F as

$$\begin{aligned} D_x F_{i,n} &= \frac{F_{i+1,n} - F_{i-1,n}}{2\triangle x} \\ D_x^2 F_{i,n} &= \frac{F_{i+1,n} - 2F_{i,n} + F_{i-1,n}}{(\triangle x)^2}. \end{aligned}$$

We employ forward difference and backward difference methods to approximate F_x and F_{xx} at the boundary of space variable x_i for $i = 1$ and I. The method is the same as shown in Equations (49)–(52).

After that we can compute a new guess on the optimal controls $\hat{c}_{i,j,n}$ and $\pi_{i,j,n}$ according to (45) and (46).

The constraints (47) are employed when deriving control variables. The investment share on stocks is binding if $\pi_S > 1$ or $\pi_S < 0$.

Given the new guess on the optimal controls, we solve again the system of equations and obtain a new guess on the value function at time t_n. We continue the iterations until the largest relative change in the value function over all i and j is below some small threshold (use tolerance at 0.1%).

Then we move to t_{n-1} until time 0. The relative change is computed as

$$RelChange = \max\left\{\max\left[\frac{\left|F^k - F^{k-1}\right|}{\max\left(10^{-5}, \left|F^{k-1}\right|\right)}\right]\right\}$$

where k is the index of iteration, $k = 1, ..., maxIter$.

Using this method, we obtain the optimal solution of $F\left(x_i, t_n\right)$, $\pi_S\left(x_i, t_n\right)$ and $\hat{c}\left(x_i, t_n\right)$ for all x_i and t_n.